

Lecture-05: Stopping Times

1 Stopping Times

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{F}_\bullet = (\mathcal{F}_t \subseteq \mathcal{F} : t \in T)$ be a filtration on this probability space for an ordered index set $T \subseteq \mathbb{R}$ considered as time.

Definition 1.1. A random variable $\tau : \Omega \rightarrow T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a *stopping time* with respect to a filtration $\mathcal{F}_\bullet$ if τ is almost surely finite and the event $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$.

Remark 1. Let $\mathcal{F}_\bullet$ be a natural filtration associated with a real-valued time-evolving random process $X : \Omega \rightarrow \mathcal{X}^T$ defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, $\mathcal{F}_t = \sigma(X_s, s \leq t)$ for all times $t \in T$.

Remark 2. A stopping time $\tau : \Omega \rightarrow T$ for the process X is an almost surely finite random variable such that if we observe the process X sequentially, then the event $\{\tau \leq t\}$ can be completely determined by the sequential observation $(X_s, s \leq t)$ until time t .

Remark 3. The intuition behind a stopping time is that its realization is determined by the past and present events but not by future events. That is, given the history of the process until time t , we can tell whether the stopping time is less than or equal to t or not. In particular, $\mathbb{E}[\mathbb{1}_{\{\tau \leq t\}} | \mathcal{F}_t] = \mathbb{1}_{\{\tau \leq t\}}$ is either one or zero.

Definition 1.2. For a process $X : \Omega \rightarrow \mathcal{X}^T$ and any Borel measurable set $A \in \mathcal{B}(\mathcal{X})$, *first hitting time* to states A by the process X is denoted by $\tau_X^A : \Omega \rightarrow T \cup \{\infty\}$, defined as $\tau_X^A \triangleq \inf \{t \in T : X_t \in A\}$.

Exercise 1.3. Let X be adapted to a filtration $\mathcal{F}_\bullet$ and $A \in \mathcal{B}(\mathcal{X})$. Show that τ_X^A is a stopping time with respect to $\mathcal{F}_\bullet$ if X is progressive and τ_X^A is almost surely finite.

Exercise 1.4. Consider an index set $T \subseteq \mathbb{R}_+$, an almost surely finite random variable $\tau : \Omega \rightarrow T \cup \{\infty\}$ taking values in a countable subset $I \subseteq T$ almost surely, and a filtration $\mathcal{F}_\bullet$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that $\{\tau \in I\} \in \mathcal{F}_0$. The random variable τ is a *stopping time* with respect to this filtration $\mathcal{F}_\bullet$ iff the event $\{\tau = t\} \in \mathcal{F}_t$ for all $t \in I$.

Theorem 1.5. Consider an almost surely finite random variable $\tau : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ and a filtration $\mathcal{F}_\bullet = (\mathcal{F}_n \subseteq \mathcal{F} : n \in \mathbb{N})$ defined on the probability space $(\Omega, \mathcal{F}, P)$. The random variable τ is a stopping time with respect to this filtration $\mathcal{F}_\bullet$ iff the event $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Proof. We first show that if $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, then τ is a stopping time. It follows from the fact that $\{\tau \leq n\} = \cup_{m \leq n} \{\tau = m\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

For the converse, we assume that τ is a stopping time and fix an $n \in \mathbb{N}$. Then $\{\tau \leq n\} \in \mathcal{F}_n$ and $\{\tau \leq n-1\} \in \mathcal{F}_n$. The result follows from the closure of an event space under complements and intersections, which implies that $\{\tau = n\} = \{\tau \leq n\} \setminus \{\tau \leq n-1\} \in \mathcal{F}_n$. $\square$

Example 1.6. Consider a random sequence $X : \Omega \rightarrow \mathcal{X}^{\mathbb{N}}$ with the natural filtration $\mathcal{F}_\bullet$ and a measurable set $A \in \mathcal{B}(\mathcal{X})$. If the first hitting time $\tau_X^A : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ for the sequence X to hit set A is almost surely finite, then τ_X^A is a stopping time. For this case, we can write $\{\tau_X^A = n\} = \cap_{k=1}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

1.1 Properties of stopping time

Lemma 1.7. Let $\tau_1, \tau_2 : \Omega \rightarrow T$ be stopping times on probability space $(\Omega, \mathcal{F}, P)$ with respect to filtration $\mathcal{F}_\bullet$. Then the following hold true.

- (a) $\min\{\tau_1, \tau_2\}$ and $\max\{\tau_1, \tau_2\}$ are stopping times.
- (b) If $P\{\tau_1 \in I\} = 1$ and $P\{\tau_2 \in I\} = 1$ for a countable $I \subseteq T$, then $\tau_1 + \tau_2$ is a stopping time.

Proof. Let $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$ be a filtration, and τ_1, τ_2 associated stopping times.

- (a) Result follows since for any $t \in T$, the event $\{\min\{\tau_1, \tau_2\} > t\} = \{\tau_1 > t\} \cap \{\tau_2 > t\} \in \mathcal{F}_t$, and the event $\{\max\{\tau_1, \tau_2\} \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$.
- (b) It suffices to show that the event $\{\tau_1 + \tau_2 \leq t\} \in \mathcal{F}_t$ for any $t \in I = \mathbb{N}$. We fix $n \in I$, and it follows from the closure of event space $\mathcal{F}_n$ under countable unions and intersection, that $\{\tau_1 + \tau_2 \leq n\} = \bigcup_{m \in \mathbb{N}} \{\tau_1 \leq n - m, \tau_2 \leq m\} \in \mathcal{F}_n$.

□

Lemma 1.8. Consider a random walk $S : \Omega \rightarrow \mathbb{R}^N$ with i.i.d. step-sizes $X : \Omega \rightarrow \mathbb{R}^N$ having finite $\mathbb{E}|X_1|$. Let $\tau : \Omega \rightarrow \mathbb{N}$ be a random variable independent of the step-size sequence such that $\mathbb{E}|\tau| < \infty$. Then,

$$\mathbb{E}S_\tau = \mathbb{E}X_1\mathbb{E}\tau.$$

Proof. Recall that the natural filtration of the random walk and the step-sizes are identical, and we denote it by $\mathcal{F}_\bullet$. We know that $P(\bigcup_{n \in \mathbb{N}} \{\tau = n\}) = 1$ and recall that conditional expectation of S_τ given the discrete random variable τ is given by $\mathbb{E}[S_\tau | \sigma(\tau)] = \sum_{n \in \mathbb{N}} \mathbb{E}[S_\tau | \tau = n] \mathbb{1}_{\{\tau=n\}}$. Since $S_n = \sum_{i=1}^n X_i$, we obtain from the tower property and linearity of conditional expectation,

$$\mathbb{E}S_\tau = \mathbb{E}[\mathbb{E}[S_\tau | \sigma(\tau)]] = \mathbb{E}\left[\sum_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E}[X_i | \tau = n] \mathbb{1}_{\{\tau=n\}}\right].$$

Since the i.i.d. random sequence X is independent of random variable τ , we get $\mathbb{E}[X_i | \tau = n] = \mathbb{E}X_1$, and it follows that $\mathbb{E}S_\tau = \mathbb{E}X_1 \mathbb{E}[\sum_{n \in \mathbb{N}} n \mathbb{1}_{\{\tau=n\}}] = \mathbb{E}X_1 \mathbb{E}\tau$. □

Lemma 1.9 (Wald). Consider a random walk $S : \Omega \rightarrow \mathbb{R}^N$ with i.i.d. step-sizes $X : \Omega \rightarrow \mathbb{R}^N$ having finite $\mathbb{E}|X_1|$. Let $\tau : \Omega \rightarrow \mathbb{N}$ be a finite mean stopping time adapted to the natural filtration $\mathcal{F}_\bullet$ of the step-size sequence X . Then,

$$\mathbb{E}S_\tau = \mathbb{E}X_1\mathbb{E}\tau.$$

Remark 4. We first examine why the proof of Lemma 1.8 breaks down for Lemma 1.9 when τ is a stopping time with respect to natural filtration of X . In the later case, it is not clear what is the value $\mathbb{E}[X_i | \tau = n]$? For example, consider the i.i.d. sequence $X \in \{0, 1\}^{\mathbb{N}}$ with $P\{X_i = 1\} = p$ and stopping $\tau \triangleq \inf\{n \in \mathbb{N} : X_i = 1\}$ adapted to natural filtration of X . In this case, for $i \leq \tau$

$$\mathbb{E}[X_i | \tau = n] = \mathbb{1}_{\{i=n\}} \neq \mathbb{E}X_i = p.$$

However, we do notice that the result somehow *magically* continues to hold, as

$$\mathbb{E}S_\tau = \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau=n\}}\right] = 1 = \mathbb{E}X_1\mathbb{E}\tau = \frac{p}{p}.$$

Proof. Recall that the filtration generated by the random walk and the step-sizes are identical, and denoted by $\mathcal{F}_\bullet$. From the independence of step sizes, it follows that X_n is independent of $\mathcal{F}_{n-1}$. Since τ is a stopping time with respect to random walk S , we observe that $\{\tau \geq n\} = \{\tau > n-1\} \in \mathcal{F}_{n-1}$, and hence it follows that random variable X_n and indicator $\mathbb{1}_{\{\tau \geq n\}}$ are independent and $\mathbb{E}[X_n \mathbb{1}_{\{\tau \geq n\}}] = \mathbb{E}X_1 \mathbb{E} \mathbb{1}_{\{\tau \geq n\}}$. Therefore,

$$\mathbb{E} \sum_{n=1}^{\tau} X_n = \mathbb{E} \sum_{n \in \mathbb{N}} X_n \mathbb{1}_{\{\tau \geq n\}} = \sum_{n \in \mathbb{N}} \mathbb{E}X_n \mathbb{E}[\mathbb{1}_{\{\tau \geq n\}}] = \mathbb{E}X_1 \mathbb{E}\left[\sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau \geq n\}}\right] = \mathbb{E}[X_1] \mathbb{E}[\tau].$$

We exchanged limit and expectation in the above step, which is not always allowed. We were able to do it by the application of dominated convergence theorem. □

1.2 Stopped σ -algebra

We wish to define an event space consisting of information of the process until a random time τ . For a stopping time $\tau : \Omega \rightarrow T$, what we want is an event space generated by the process until the stopping time τ , i.e. something like $\sigma(X_s : s \leq \tau)$. However, this definition doesn't make sense, since the random time τ is a random variable itself. We do note that the event $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in T$, since τ is a stopping time. Hence, what makes sense is the set of all events whose intersection with event $\{\tau \leq t\}$ belongs to the event subspace $\mathcal{F}_t$ for all $t \geq 0$.

Definition 1.10. For a stopping time τ adapted to the filtration $\mathcal{F}_\bullet \triangleq (\mathcal{F}_t : t \in T)$, the *stopped σ -algebra* is

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in T\}.$$

Proposition 1.11. The collection of events $\mathcal{F}_\tau$ is a σ -algebra.

Proof. It suffices to verify the following three conditions.

- (a) Since τ is a stopping time, it follows that $\Omega \in \mathcal{F}_\tau$.
- (b) Let $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ and we can write $A^c \cap \{\tau \leq t\} = \{\tau \leq t\} \setminus (A \cap \{\tau \leq t\}) \in \mathcal{F}_t$.
- (c) From closure of $\mathcal{F}_t$ under countable unions, it follows that $\mathcal{F}_\tau$ is closed under countable unions. $\square$

Remark 5. Informally, the event space $\mathcal{F}_\tau$ has information up to the random time τ . That is, it is a collection of measurable sets that are determined by the process until time τ .

Remark 6. Any measurable set $A \in \mathcal{F}$ can be written as $A = (A \cap \{\tau \leq t\}) \cup (A \cap \{\tau > t\})$. All such sets A such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$ is a member of the stopped σ -algebra. We note that any event $A \in \mathcal{F}_\tau$ does not guarantee that $A \cap \{\tau > t\} \in \mathcal{F}_t$. Otherwise, $\mathcal{F}_\tau = \mathcal{F}$.

Lemma 1.12. Consider a random variable $Y : \Omega \rightarrow \mathbb{R}$, and a stopping time $\tau : \Omega \rightarrow T \subseteq \mathbb{R}_+$ with respect to filtration $\mathcal{F}_\bullet$ defined on probability space $(\Omega, \mathcal{F}, P)$. Then Y is $\mathcal{F}_\tau$ measurable if and only if $Y \mathbb{1}_{\{\tau \leq t\}}$ is $\mathcal{F}_t$ measurable for all $t \in T$.

Proof. The collection of events $\{Y^{-1}(-\infty, y] \cap \tau^{-1}(-\infty, t] : y \in \mathbb{R}\}$ generate the event space generated by the random variable $Y \mathbb{1}_{\{\tau \leq t\}}$. It follows that Y is $\mathcal{F}_\tau$ measurable if and only if $\{Y \leq y\} \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}$. $\square$

Definition 1.13. Consider a process $X : \Omega \rightarrow \mathcal{X}^T$ adapted to a filtration $\mathcal{F}_\bullet$, and a stopping time $\tau : \Omega \rightarrow T$ for the process X , then the stopped process X^τ is defined as $X_t^\tau \triangleq X_{t \wedge \tau}$ for all $t \in T$.

Lemma 1.14. If $X : \Omega \rightarrow \mathcal{X}^T$ is progressively measurable, then the stopped process X^τ is also progressively measurable and adapted to the same filtration.

Proof. We observe that $(X^\tau)^{-1}(A) \cap ((-\infty, s] \times \Omega) = \{(u, \omega) : u \leq \tau, u \leq s, X_u \in A\} = (X^{\tau \wedge s})^{-1}(A) \in \mathcal{B}((-\infty, s]) \otimes \mathcal{F}_s$ for any Borel measurable set $A \in \mathcal{B}(\mathbb{R})$ and $s \in T$. $\square$

Lemma 1.15. Let τ, τ_1, τ_2 be stopping times, and $X : \Omega \rightarrow \mathcal{X}^T$ a random process, all adapted to a filtration $\mathcal{F}_\bullet$. Then, the following are true.

- (a) If $\tau_1 \leq \tau_2$ almost surely, then $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.
- (b) $\sigma(\tau) \subseteq \mathcal{F}_\tau$.
- (c) If X is progressively measurable, then $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$.

Proof. Recall, that for any $t \geq 0$, we have $\{\tau \leq t\} \in \mathcal{F}_t$.

- (a) From the hypothesis $\tau_1 \leq \tau_2$ a.s., we get $\{\tau_2 \leq t\} \subseteq \{\tau_1 \leq t\}$ a.s., where both events belong to $\mathcal{F}_t$ since they are stopping times. The result follows since for any $A \in \mathcal{F}_{\tau_1}$ and $t \in T$, we can write $A \cap \{\tau_2 \leq t\} = A \cap \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$.
- (b) Any event $A \in \sigma(\tau)$ is generated by inverse images $\{\tau \leq s\}$ for $s \in \mathbb{R}$. Indeed $\{\tau \leq s\} \in \mathcal{F}_\tau$ since $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\} \in \mathcal{F}_t$, for all $t \in T$.
- (c) If X is progressive, then so is the stopped process X^τ and adapted to the same filtration $\mathcal{F}_\bullet$ from Lemma 1.14. It follows that $X_\tau \mathbb{1}_{\{\tau \leq t\}}$ is $\mathcal{F}_t$ measurable for all $t \in T$, and hence X_τ is $\mathcal{F}_\tau$ measurable. $\square$

Theorem 1.16. Let $X : \Omega \rightarrow \mathcal{X}^T$ be a random process and $\tau : \Omega \rightarrow T$ be a stopping time both adapted to filtration $\mathcal{F}_\bullet$. If X is progressive then for the stopped process X^τ , we have $\sigma(X^\tau) \subseteq \mathcal{F}_\tau$.

Proof. From Lemma 1.14, it follows that the stopped process X^τ is progressive. We fix $s \in T$, then it suffices to show that $\sigma(X_s^\tau) \subseteq \mathcal{F}_\tau$. To this end, we take $B \in \mathcal{B}(\mathcal{X})$, $t \in T$, and observe that

$$(X_s^\tau)^{-1}(B) \cap \{\tau \leq t\} = X_{s \wedge \tau}^{-1}(B) \cap \{\tau \leq t\} = (X_\tau^{-1}(B) \cap \{\tau \leq s \wedge t\}) \cup (X_s^{-1}(B) \cap \{s < \tau \leq t\}).$$

From the progressive measurability of X , we have $\sigma(X_\tau) \subseteq \mathcal{F}_\tau$ and hence $X_\tau^{-1}(B) \cap \{\tau \leq s \wedge t\} \in \mathcal{F}_{s \wedge t} \subseteq \mathcal{F}_t$. Further, $X_s^{-1}(B) \cap \{s < \tau \leq t\} \in \mathcal{F}_t$. $\square$