Lecture-06: Strong Markov Property

1 Strong Markov property

We will consider real valued processes $X : \Omega \to \mathcal{X}^T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with state space $\mathcal{X} \subseteq \mathbb{R}$ and ordered index set $T \subseteq \mathbb{R}$, adapted to its natural filtration by $\mathcal{F}_\bullet = (\mathcal{F}_t : t \in T)$, where $\mathcal{F}_t \triangleq \sigma(X_s, s \leqslant t)$ for all $t \in T$.

Definition 1.1. A process $X : \Omega \to \mathcal{X}^T$ adapted to its natural filtration $\mathcal{F}_\bullet$, is called **Markov** if for each $t \geqslant s$

$$\mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \sigma(X_s)].$$

Example 1.2. An independent process is trivially Markov, since

$$\mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \mathcal{F}_s] = \mathbb{E}\mathbb{1}_{\{X_t \leqslant x\}} = \mathbb{E}[\mathbb{1}_{\{X_t \leqslant x\}} \mid \sigma(X_s)].$$

Example 1.3. Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined in term of independent step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ as $S_n \triangleq \sum_{i=1}^n X_i$ for all $n \in \mathbb{N}$. The random walk S is Markov with respect to its natural filtration $\mathcal{F}_\bullet$. To see this, we take $n \in \mathbb{N}$, denote the distribution function for X_{n+1} as $F_{X_{n+1}} : \mathbb{R} \to [0,1]$, and observe from the independence of X_{n+1} and $\mathcal{F}_n$ that

$$\mathbb{E}[\mathbb{1}_{\{S_{n+1} \leqslant x\}} \mid \mathcal{F}_n] = F_{X_{n+1}}(x - S_n) = \mathbb{E}[\mathbb{1}_{\{X_{n+1} \leqslant x - S_n\}} \mid \sigma(S_n)] = \mathbb{E}[\mathbb{1}_{\{S_{n+1} \leqslant x\}} \mid \sigma(S_n)].$$

Definition 1.4. Let $X : \Omega \to \mathcal{X}^T$ be a real valued Markov process adapted to its natural filtration $\mathcal{F}_\bullet$. Let τ be a stopping time with respect to this filtration, then the process X is called **strongly Markov** at τ if for all $x \in \mathbb{R}$ and $t > 0$, we have

$$\mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \mathcal{F}_\tau] = \mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \sigma(X_\tau)]. \tag{1}$$

Exercise 1.5. Consider a random process $X : \Omega \to \mathcal{X}^{\mathbb{R}}$ adapted to its natural filtration $\mathcal{F}_\bullet$, a stopping time $\tau : \Omega \to I \subseteq \mathbb{R}$ adapted to $\mathcal{F}_\bullet$ and a random variable $Y : \Omega \to \mathbb{R}$ all defined on the same probability space $(\Omega, \mathcal{F}, P)$. If I is countable, then show that

$$\mathbb{E}[Y \mid \sigma(X_\tau)] = \sum_{i \in I} \mathbb{1}_{\{\tau = i\}} \mathbb{E}[Y \mid \sigma(X_i, \{\tau = i\})].$$

Lemma 1.6. *Consider a Markov process $X : \Omega \to \mathcal{X}^T$ adapted to its natural filtration $\mathcal{F}_\bullet$. and a stopping time τ with respect to $\mathcal{F}_\bullet$. If the stopping time τ is almost surely countable, then the process X is strongly Markov at τ.*

Proof. Let $I \subseteq T$ be the countable set such that $P\{\tau \notin I\} = 0$. We will show that the right hand side of (1) satisfies measurability, integrability, and orthogonality of conditional expectation $\mathbb{E}[\mathbb{1}_{\{X_{t+\tau} \leqslant x\}} \mid \mathcal{F}_\tau]$, and the result follows from the a.s. uniqueness of conditional expectation.

Measurability. Recall $\sigma(X_\tau) \subseteq \sigma(X^\tau) \subseteq \mathcal{F}_\tau$, and since the conditional expectation $\mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \mid \sigma(X_\tau)]$ is $\sigma(X_\tau)$ measurable, it is $\mathcal{F}_\tau$ measurable.

Integrability. Since $0 \leqslant \mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \leqslant 1$, from the monotonicity of the conditional expectation it follows that $0 \leqslant \mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \mid \sigma(X_\tau)] \leqslant 1$, and hence it is absolutely integrable.

Orthogonality. Fix $A \in \mathcal{F}_\tau$. Since I is countable, we have $A \cap \{\tau = i\} \in \mathcal{F}_i$ for each $i \in I$. Further, we can write $A = \cup_{i \in I} A \cap \{\tau = i\}$ from almost sure countability of τ. It suffices to show that for all $x \in \mathbb{R}$ and $t > 0$,

$$\mathbb{E}[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\{X_{\tau+t} \leqslant x\}} \mid \sigma(X_\tau)]] = \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{\tau+t} \leqslant x\}}].$$

From the almost sure sum $1 = \sum \mathbb{1}_{\{\tau=i\}}$, linearity of expectation, the tower property of conditional expectation, $\mathcal{F}_i$-measurability of $A \cap \{\tau = i\}$, and pulling out what's known, we obtain

$$\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{X_{t+\tau} \leqslant x\}}] = \sum_{i\in I} \mathbb{E}[\mathbb{1}_{A\cap\{X_{t+\tau}\leqslant x\}\cap\{\tau=i\}}] = \sum_{i\in I} \mathbb{E}[\mathbb{E}[\mathbb{1}_{A\cap\{X_{t+i}\leqslant x\}\cap\{\tau=i\}}|\mathcal{F}_i]] = \mathbb{E}[\mathbb{1}_A \sum_{i\in I} \mathbb{1}_{\{\tau=i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+i}\leqslant x\}}|\mathcal{F}_i]].$$

From Markov property of process X, we have $\mathbb{E}[\mathbb{1}_{\{X_{t+i}\leqslant x\}} \mid \mathcal{F}_i] = \mathbb{E}[\mathbb{1}_{\{X_{t+i}\leqslant x\}} \mid \sigma(X_i)]$. This result together with Markovity of X, Exercise 1.5, and pulling out what's known, we obtain

$$\sum_{i\in I} \mathbb{1}_{\{\tau=i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+i}\leqslant x\}} \mid \sigma(X_i)] = \sum_{i\in I} \mathbb{1}_{\{\tau=i\}} \mathbb{E}[\mathbb{1}_{\{X_{t+\tau}\leqslant x\}} \mid \sigma(X_i, \{\tau = i\})] = \mathbb{E}[\mathbb{1}_{\{X_{t+\tau}\leqslant x\}} \mid \sigma(X_\tau)].$$

The result follows from multiplying both sides of the above equation by $\mathbb{1}_A$ and taking expectation on both sides.

□

Corollary 1.7. *Any Markov process on countable index set T is strongly Markov.*

Proof. For a countable index set T, any associated stopping time is countable. □

Corollary 1.8. *Consider an* i.i.d. *random sequence $X : \Omega \to \mathcal{X}^{\mathbb{N}}$ with its natural filtration denoted by $\mathcal{F}_\bullet$, and a stopping time $\tau : \Omega \to \mathbb{N}$ adapted to $\mathcal{F}_\bullet$. Then for each $n \in \mathbb{N}$, the random vector $(X_{\tau+1}, \dots, X_{\tau+n})$ is independent of $\mathcal{F}_\tau$ and identically distributed to $(X_1, \dots, X_n)$.*

Proof. Let $F : \mathbb{R} \to [0,1]$ be the common distribution for the *i.i.d.* sequence X, then it suffices to show that

$$\mathbb{E}\left[\prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i}\leqslant x_i\}} \mid \mathcal{F}_\tau\right] = \prod_{i=1}^{n} F(x_i), \quad x \in \mathbb{R}^n.$$

Since RHS of the above equation is a constant in $[0,1]$, the measurability and integrability are clear. To show orthogonality, we fix $A \in \mathcal{F}_\tau$ and we need to show that

$$\mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} F(x_i)] = \mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i}\leqslant x_i\}}].$$

We can write $\mathbb{1}_A = \sum_{m\in\mathbb{N}} \mathbb{1}_A \mathbb{1}_{\{\tau=m\}}$ where $A \cap \{\tau = m\} \in \mathcal{F}_m$. Therefore, from the linearity of expectation, the tower property of conditional expectation, and from X being *i.i.d.* , we can write

$$\mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} \mathbb{1}_{\{X_{\tau+i}\leqslant x_i\}}] = \sum_{m\in\mathbb{N}} \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau=m\}} \mathbb{E}[\prod_{i=1}^{n} \mathbb{1}_{\{X_{m+i}\leqslant x_i\}} \mid \mathcal{F}_m]] = \mathbb{E}[\mathbb{1}_A \sum_{m\in\mathbb{N}} \mathbb{1}_{\{\tau=m\}} \prod_{i=1}^{n} F(x_i)] = \mathbb{E}[\mathbb{1}_A \prod_{i=1}^{n} F(x_i)].$$

□

Theorem 1.9. *Let $X : \Omega \to \mathcal{X}^T$ be any real-valued Markov process adapted to its natural filtration $\mathcal{F}_\bullet$, with right-continuous sample paths. If the map $t \mapsto \mathbb{E}[f(X_s) \mid \sigma(X_t)]$ is right-continuous for each bounded continuous function f, then X is strongly Markov.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function, $t \geqslant 0$, and τ be an $\mathcal{F}_\bullet$-adapted stopping time. It suffices to show that $f(X_t)$ satisfies the strong Markov property. For each $m \in \mathbb{N}$, consider the intervals $I_{k,m} \triangleq ((k-1)2^{-m}, k2^{-m}]$ for all $k \in [2^{2m}]$, and define

$$\tau_m \triangleq \sum_{k=1}^{2^{2m}} k2^{-m} \mathbb{1}_{\{\tau\in I_{k,m}\}}.$$

We observe that τ_m is adapted to $\mathcal{F}_\bullet$ and takes countable values for each m. Further, we have $\tau \mathbb{1}_{\{\tau\leqslant 2^m\}} \leqslant \tau_m \leqslant 2^m$ and τ_m is eventually decreasing in m for each outcome ω. From a.s. finiteness of stopping time τ, for almost all outcomes $\omega \in \Omega$ there exists an $m_0(\omega) \in \mathbb{N}$ such that $\tau(\omega) \leqslant \tau_m(\omega)$ for all $m \geqslant m_0(\omega)$. Hence, $\tau_m \downarrow \tau$ almost surely. Since $\tau \leqslant \tau_m$, it follows that $\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_m}$. From the strong Markov property for the Markov process X at countably valued stopping times, we have

$$\mathbb{E}[f(X_{\tau_m+t}) \mid \mathcal{F}_{\tau_m}] = \mathbb{E}[f(X_{\tau_m+t}) \mid \sigma(X_{\tau_m})].$$

From the orthogonality property of conditional expectation, it follows that for each $A \in \mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_m}$, we have

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau_m+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau_m+t})|\sigma(X_{\tau_m})]].$$

Taking limit as $\tau_m \downarrow \tau$ on both sides and applying dominated convergence theorem, we get

$$\mathbb{E}[\mathbb{1}_A f(X_{\tau+t})] = \mathbb{E}[\mathbb{1}_A \mathbb{E}[f(X_{\tau+t})|\sigma(X_\tau)]].$$

□