Lecture-09: Limit Theorems

1 Growth of renewal counting processes

We consider a renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ with natural filtration denoted by $\mathcal{F}_\bullet \triangleq (\mathcal{F}_n : n \in \mathbb{N})$ and *i.i.d.* inter renewal time sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $S_n \triangleq \sum_{i=1}^n X_i$ and $\mathcal{F}_n \triangleq \sigma(X_1, \dots, X_n)$ for each $n \in \mathbb{N}$. The associated renewal counting process $N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ and the renewal function $m \in \mathbb{R}_+^{\mathbb{R}_+}$ are defined as $N_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{S_n \leqslant t\}}$ and $m_t \triangleq \mathbb{E}N_t$ for each $t \in \mathbb{R}_+$.

Lemma 1.1. *If first renewal instant X_1 has finite mean, then $N_\infty \triangleq \lim_{t\to\infty} N_t$ is almost surely infinite.*

Proof. Since $X_n \geqslant 0$ and $\mathbb{E}[X_n] < \infty$, it follows that $P\{X_n = \infty\} = 0$. Further, we observe that

$$P\{N_\infty < \infty\} = P\Big(\bigcup_{n\in\mathbb{N}} \{N_\infty < n\}\Big) = P\Big(\bigcup_{n\in\mathbb{N}} \{S_n = \infty\}\Big) = P\Big(\bigcup_{n\in\mathbb{N}} \{X_n = \infty\}\Big) \leqslant \sum_{n\in\mathbb{N}} P\{X_n = \infty\} = 0.$$

□

Corollary 1.2. *If the first renewal instant and subsequent inter renewal times have finite means for a delayed renewal process N^D, then $P\left\{\lim_{t\to\infty} N_t^D = \infty\right\} = 1$.*

Proof. For each $n \in \mathbb{N}$, the inter renewal time X_n is positive and has finite mean, and hence is finite almost surely. Therefore, $P\{N_\infty < \infty\} \leqslant \sum_{n\in\mathbb{N}} P\{X_n = \infty\} = 0$. □

1.1 Growth of counting process

We observed that the number of renewals N_t increases to infinity with the length of the duration t. We will show that the growth of N_t is asymptotically linear with time t, and we will find this coefficient of linear growth of N_t with time t.

Theorem 1.3 (Strong law for renewal process). *For a renewal counting process N with* i.i.d. *inter renewal time sequence X, $\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}X_1}$ almost surely.*

Proof. Recall that $X_1 \geqslant 0$ and we consider two cases.

Case $\mathbb{E}X_1 = \infty$. It follows that $\alpha \triangleq P\{X_1 < \infty\} < 1$ and hence $P\{N_\infty = n\} = \alpha^n(1-\alpha)$ for all $n \in \mathbb{Z}_+$. It follows that $\cup_{n\in\mathbb{Z}_+} \{N_\infty = n\} = \cup_{n\in\mathbb{Z}_+} \{N_\infty \leqslant n\}$ is an almost sure event, and hence $\lim_{t\to\infty} N_t/t = 0$ almost surely.

Case $\mathbb{E}X_1 < \infty$. For this case, $\lim_{t\to\infty} N_t = \infty$ almost surely from Lemma 1.1. We note that $S_{N_t} \leqslant t < S_{N_t+1}$, and dividing by N_t, we get

$$\sum_{n\in\mathbb{Z}_+} \frac{S_n}{n} \mathbb{1}_{\{N_t = n\}} = \frac{S_{N_t}}{N_t} \leqslant \frac{t}{N_t} < \frac{S_{N_t+1}}{N_t} = \sum_{n\in\mathbb{Z}_+} \frac{S_{n+1}}{n} \mathbb{1}_{\{N_t = n\}}.$$

Since $\lim_{t\to\infty} N_t = \infty$ almost surely, we have $\lim_{t\to\infty} \frac{S_{N_t}}{N_t} = \lim_{n\to\infty} \frac{S_n}{n}$ and $\lim_{t\to\infty} \frac{S_{N_t+1}}{N_t} = \lim_{n\to\infty} \frac{S_{n+1}}{n}$ almost surely. The result follows from L^1 strong law of large numbers.

□

Corollary 1.4. *For a delayed renewal process N^D, $\lim_{t\to\infty} \frac{N_t^D}{t} = \frac{1}{\mathbb{E}X_2}$ almost surely.*

Proof. From L^1 strong law of large numbers, we observe that $\lim_{n\in\mathbb{N}} S_n/n = \mathbb{E}X_2$, and the result follows.

□

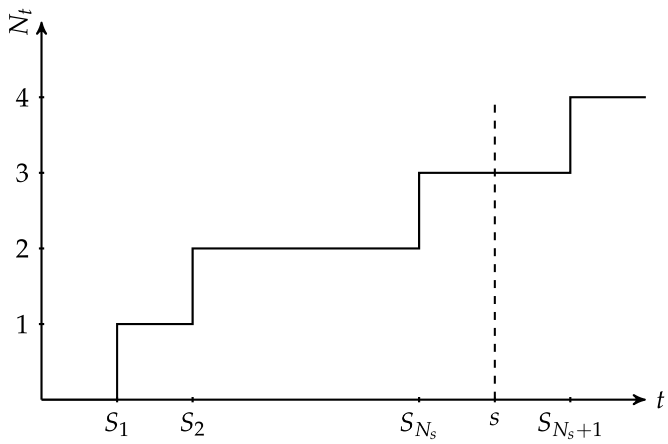

Figure 1: Time of last renewal

Example 1.5. Suppose, you are in a casino with infinitely many games. We assume that $X : \Omega \to [0,1]^{\mathbb{N}}$ is an *i.i.d.* uniform sequence where X_i is the random probability of win in the game $i \in \mathbb{N}$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins. Let $N(n)$ denote the number of losses in n plays. Then the fraction of wins $P_W(n)$ is given by $P_W(n) = \frac{n-N(n)}{n}$. We pick a strategy where any game is selected to play, and continue to be played till the first loss. We show that $\lim_{n\to\infty} P_W(n) = 1$ for this proposed strategy. Let T_i be the number of times a game i is played. We observe that the conditional probability mass function for the number of plays for each game i is geometrically distributed as

$$\mathbb{E}[\mathbb{1}_{\{T_i=k\}} \mid \sigma(X_i)] = X_i^{k-1}(1-X_i), \quad k \in \mathbb{N}.$$

Hence, it follows that T_i are *i.i.d.* random variables with mean $\mathbb{E}T_i = \mathbb{E}[\mathbb{E}[T_i \mid X_i]] = \mathbb{E}\left[\frac{1}{1-X_i}\right] = \infty$. It follows that each loss is a renewal event, and from the strong law of renewal process, we obtain

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}[\text{Time till first loss}]} = \frac{1}{\mathbb{E}T_i} = 0.$$

1.2 Growth of renewal function

Basic renewal theorem implies N_t/t converges to $1/\mathbb{E}X_1$ almost surely. We are next interested in convergence of the ratio m_t/t. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. To illustrate this, we have the following example.

Example 1.6. Consider a Bernoulli random sequence $X : \Omega \to \{0,1\}^{\mathbb{N}}$ with probability $P\{X_n = 1\} = \frac{1}{n}$, and another random sequence $Y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ defined as $Y_n \triangleq nX_n$ for $n \in \mathbb{N}$. Then, $P\{Y_n = 0\} = 1 - \frac{1}{n}$. That is $Y_n \to 0$ a.s. However, $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$. In this case, $\lim_{n\in\mathbb{N}} \mathbb{E}[Y_n] = 1$.

Even though, basic renewal theorem does *NOT* imply it, we show that m_t/t converging to $1/\mathbb{E}X_1$.

Proposition 1.7 (Wald's Lemma for renewal process). *If $\mathbb{E}X_1 \in (0,\infty)$, then $N_t + 1$ is a stopping time adapted to $\mathcal{F}_\bullet$, and $\mathbb{E}\sum_{i=1}^{N_t+1} X_i = (1+m_t)\mathbb{E}X_1$.*

Proof. Recall that N_t is almost surely finite if $\mathbb{E}X_1 > 0$. We fix $n \in \mathbb{N}$, and observe that

$$\{N_t + 1 = n\} = \{S_{n-1} \leqslant t < S_n\} \in \sigma(X_1, \dots, X_n) = \mathcal{F}_n.$$

Thus $N_t + 1$ is a stopping time adapted to $\mathcal{F}_\bullet$, and the result follows from Wald's Lemma. $\square$

Theorem 1.8 (Elementary renewal theorem). *For a renewal counting process* $\lim_{t\to\infty} \frac{m_t}{t} = \frac{1}{\mathbb{E}X_1}$.

Proof. Recall that $X_1 \geqslant 0$. We first show two limiting bounds on $\frac{m_t}{t}$.

Lower bound. Taking expectations on both sides of $S_{N_t+1} > t$ and using Wald's Lemma for renewal processes in Proposition 1.7, we have $\mathbb{E}X_1(m_t+1) > t$. Dividing both sides by $t\mathbb{E}X_1$ and taking $\liminf$ on both sides, we get $\liminf_{t\to\infty} \frac{m_t}{t} \geqslant \frac{1}{\mathbb{E}X_1}$.

Upper bound. We employ a truncated random variable argument. We define truncated inter renewal times $\bar{X} : \Omega \to [0, M]^{\mathbb{N}}$ defined as $\bar{X}_n \triangleq X_n \wedge M$ for each $n \in \mathbb{N}$, and with common mean denoted by $\mu_M \triangleq \mathbb{E}(X_1 \wedge M)$. Since X is *i.i.d.* , so is the truncated sequence $\bar{X}$, and hence we can define the corresponding renewal sequence $\bar{S} : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ as $\bar{S}_n \triangleq \sum_{i=1}^n \bar{X}_i$ for each $n \in \mathbb{N}$, and the counting process $\bar{N} : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$ as $\bar{N}_t \triangleq \sum_{n\in\mathbb{N}} \mathbb{1}_{\{\bar{S}_n \leqslant t\}}$ for each $t \in \mathbb{R}_+$. Note that since $S_n \geqslant \bar{S}_n$, the number of arrivals would be higher for renewal process $\bar{N}_t$ with truncated random variables. That is, $N_t \leqslant \bar{N}_t$, and hence $m_t \leqslant \bar{m}_t$ from the monotonicity of expectation. Further, due to truncation of inter arrival time, next renewal happens within M units of time, that is $\bar{S}_{\bar{N}_t+1} \leqslant t + M$. From the monotonicity of expectation and Wald's Lemma for renewal processes in Proposition 1.7, we get $(1 + \bar{m}_t)\mu_M \leqslant t + M$. Dividing both sides by $t\mu_M$ and the fact that $m_t \leqslant \bar{m}_t$ for all times $t \in \mathbb{R}_+$, we obtain $\limsup_{t\to\infty} \frac{m_t}{t} \leqslant \limsup_{t\to\infty} \frac{\bar{m}_t}{t} \leqslant \frac{1}{\mu_M}$. Since the choice of M was arbitrary, we can take $M \to \infty$ so that $\mu_M \to \mathbb{E}X_1$.

We consider two cases for mean $\mathbb{E}X_1$.

Case $\mathbb{E}X_1 = 0$. From lower bound $\liminf_{t\to\infty} \frac{m_t}{t} \geqslant \frac{1}{\mathbb{E}X_1}$ and positivity of m_t and t, the result follows.

Case $\mathbb{E}X_1 = \infty$. From upper bound $\limsup_{t\to\infty} \frac{m_t}{t} \leqslant \frac{1}{\mathbb{E}X_1}$.

Case $\mathbb{E}X_1 \in (0, \infty)$. The result follows by combining the upper and lower bounds.

□

Corollary 1.9. *If a delayed renewal process has finite inter renewal durations, then* $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mathbb{E}X_2}$.

Example 1.10 (Markov chain). Consider a positive recurrent discrete time Markov chain $X : \Omega \to \mathcal{X}^{\mathbb{Z}_+}$. Let the initial state be $X_0 = x \in \mathcal{X}$ and $\tau_y^+(0) = 0$ for $y \neq x \in \mathcal{X}$, then we can inductively define the kth recurrent time to state y as a stopping time

$$\tau_y^+(k) \triangleq \inf\left\{n > \tau_y^+(k-1) : X_n = y\right\}.$$

Since any discrete time Markov chain satisfies the strong Markov property, it follows that the sequence of stopping times $\tau_y^+ : \Omega \to \mathbb{N}^{\mathbb{N}}$ is a delayed renewal sequence, where the distribution of first renewal time is $P_x\left\{\tau_y^+(1) = k\right\} = f_{xy}^{(k)}$ and the common distribution of subsequent inter renewal durations is $P_y\left\{\tau_y^+(1) = k\right\} = f_{yy}^{(k)}$ for $k \in \mathbb{N}$. We denote the associated counting process by $N_y : \Omega \to \mathbb{Z}_+^{\mathbb{N}}$, where $N_y(n)$ denotes the number of visits to state y up to time n, and is defined as

$$N_y(n) \triangleq \sum_{k\in\mathbb{N}} \mathbb{1}_{\left\{\tau_y^+(k) \leqslant n\right\}} = \sum_{k=1}^{n} \mathbb{1}_{\{X_k = y\}}.$$

From the strong law for delayed renewal sequence τ_y^+ with finite mean inter renewal time $\mu_{yy} \triangleq \mathbb{E}_y\tau_y^+(1)$ (which is also the mean recurrence time to state y), it follows that

$$P_y\left\{\lim_{n\in\mathbb{N}} \frac{N_y(n)}{n} = \frac{1}{\mu_{yy}}\right\} = 1.$$

From the elementary renewal theorem for delayed renewal sequence τ_y^+, it follows that

$$\lim_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} p_{xy}^{(k)} = \lim_{n\in\mathbb{N}} \frac{1}{n}\mathbb{E}_x[N_y(n)] = \frac{1}{\mu_{yy}}.$$

1.3 Central limit theorem for renewal processes

Theorem 1.11. *If $\mu \triangleq \mathbb{E}X_1$ and $\sigma^2 \triangleq \mathrm{Var}(X_1)$ such that $\mu, \sigma^2 \in (0, \infty)$, then $N_t \to \mathcal{N}(\frac{t}{\mu}, \sigma^2 \frac{t}{\mu^3})$ for large t in distribution. Specifically,* $\lim_{t\to\infty} P\left\{\frac{N_t - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx.$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N_t < u\} = \{S_u > t\}$. By equating probability measures on both sides, we get

$$P\{N_t < u\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t - u\mu}{\sigma\sqrt{u}}\right\} = P\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-1/2}\right\}.$$

By central limit theorem, $\frac{S_u - u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. We also observe that $\lim_{t\to\infty} -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^{-1/2} = -y$. These results combine with the symmetry of normal random variable to give us the result. $\square$