

Lecture-11: Key Renewal Theorem

1 Blackwell Theorem

Lemma 1.1. Consider a renewal sequence S with inter renewal time distribution F and renewal function m . If $\inf \{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$, then $\sup \{m_t - m_{t-b} : t \in \mathbb{R}_+\} < \infty$ for any $b > 0$.

Proof. Recall that $m = \sum_{n \in \mathbb{N}} F_n$ and hence $m * F = m - F$. This implies that $m * (1 - F) = F$. Since the function $1 - F$ is monotonically non increasing, $\inf_{s \in [0, b]} \bar{F}(s) = \bar{F}(b)$. Therefore,

$$1 \geq F(t) = \int_0^t dm_s \bar{F}(t-s) \geq \int_{t-b}^t dm_s \bar{F}(t-s) \geq (m_t - m_{t-b}) \bar{F}(b).$$

Since $F(b) < 1$ for any $b \in \mathbb{R}_+$, we obtain the result. $\square$

Example 1.2. Consider a continuous nondecreasing map $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(a+b) = g(a) + g(b)$ for each $a, b \in \mathbb{R}$, then $g(a) = ca$ for some $c \geq 0$ and all $a \in \mathbb{R}$. We show this by induction that $g(m) = mg(1)$ and $g(\frac{m}{n}) = g(1)\frac{m}{n}$, for any $m, n \in \mathbb{Z}$ and $n \neq 0$. It follows that for any rational $q \in \mathbb{Q}$ we have $g(q) = g(1)q$, and from continuity of g and taking limit of rationals approaching a real number, we obtain that for any real $a \in \mathbb{R}$, we have $g(a) = g(1)a$. We define $c \triangleq g(1)$ which is nonnegative since g is nondecreasing.

Theorem 1.3 (Blackwell's Theorem). Consider a renewal sequence S with renewal function m , and i.i.d. inter renewal time sequence X with common distribution F such that $\inf \{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$.

- (a) If the renewal sequence is aperiodic, then $\lim_{t \rightarrow \infty} m_{t+a} - m_t = a/\mathbb{E}X_1$ for all $a \geq 0$.
- (b) If the renewal sequence has period d , then $\lim_{n \rightarrow \infty} m_{(n+1)d} - m_{nd} = d/\mathbb{E}X_1$.

Proof. We will show later that the following limit exists for aperiodic renewal sequences

$$g(a) \triangleq \lim_{t \rightarrow \infty} [m_{t+a} - m_t] \quad (1)$$

- (a) However, we show that if this limit does exist, it is equal to $a/\mathbb{E}X_1$ as a consequence of elementary renewal theorem. To this end, note that $m_{t+a+b} - m_t = m_{t+a+b} - m_{t+a} + m_{t+a} - m_t$. Taking limits on both sides of the above equation, we conclude that $g(a+b) = g(a) + g(b)$. The only nondecreasing solution of such a $g \in \mathbb{R}_+^{\mathbb{R}}$ is $g(a) = ca$, for all $a \in \mathbb{R}_+$ and some positive constant c . To show $c = 1/\mathbb{E}X_1$, we define a sequence $x \in \mathbb{R}_+^{\mathbb{N}}$ in terms of renewal function m_t for each $n \in \mathbb{N}$, as

$$x_n \triangleq m_n - m_{n-1}.$$

Note that $\sum_{i=1}^n x_i = m_n$ and $\lim_{n \in \mathbb{N}} x_n = g(1) = c$. Further recall that, if a sequence $x \in \mathbb{R}^{\mathbb{N}}$ converges, then the running average sequence $a \in \mathbb{R}^{\mathbb{N}}$ defined as $a_n \triangleq \frac{1}{n} \sum_{i=1}^n x_i$ converges to the same limit. Hence, we have the Cesàro mean converging to $\lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=1}^n x_i = \lim_{n \in \mathbb{N}} \frac{m_n}{n} = c$. Therefore, we can conclude $c = 1/\mathbb{E}X_1$ by elementary renewal theorem.

- (b) If the renewal sequence is periodic with period d , the limit in (1) doesn't exist, as shown in the following example. However, the theorem is true for lattice again by elementary renewal theorem. We can define $x_n \triangleq m_{nd} - m_{(n-1)d}$ and observe that $\sum_{i=1}^n x_i = m_{nd}$ and the Cesàro mean $\frac{1}{n} \sum_{i=1}^n x_i$ converges to $d/\mathbb{E}X_1$ by elementary renewal theorem.

□

Example 1.4. Consider a renewal sequence with $P\{X_1 = 1\} = 1$, that is, there is a renewal at every positive integer time instant with unit probability. We observe that it is a periodic renewal sequence with period $d = 1$. Now, for $a = 0.25$, and $t_n = n + (-1)^n a$, we see that $m_{t_n} = N_{t_n} = n - \mathbb{1}_{\{n \text{ odd}\}}$, and $m_{t_n+a} = n$. It follows that $m_{t_n+a} - m_{t_n} = \mathbb{1}_{\{n \text{ odd}\}}$, and hence $\lim_{t \rightarrow \infty} m_{t+a} - m_t$ does not exist. It follows that $\lim_{t \rightarrow \infty} m_{t+a} - m_t$ does not exist.

Exercise 1.5. Let m be the renewal function associated with an aperiodic renewal sequence. Show that the following limit exists

$$g(a) \triangleq \lim_{t \rightarrow \infty} [m_{t+a} - m_t].$$

Remark 1. For a renewal sequence with positive periodicity $d > 0$, there can be no more than one renewal at each time instant nd . In this case,

$$\lim_{n \rightarrow \infty} P\{\text{renewal at } nd\} = d/\mathbb{E}X_1.$$

Corollary 1.6 (Delayed Blackwell's Theorem). Consider an aperiodic and recurrent delayed renewal process S^D with independent inter renewal times X with first inter renewal time distribution G and common inter renewal time distribution F for $(X_n : n \geq 2)$ such that $\inf\{x \in \mathbb{R}_+ : F(x) = 1\} = \infty$. We denote the associated renewal function as m^D .

- (a) If the renewal sequence is aperiodic, then $\lim_{t \rightarrow \infty} m_{t+a}^D - m_t^D = a/\mathbb{E}X_2$ for all $a \geq 0$.
- (b) If renewal sequence has period d , then $\lim_{n \rightarrow \infty} m_{(n+1)d}^D - m_{nd}^D = d/\mathbb{E}X_2$.

2 Key Renewal Theorem

Theorem 2.1 (Key renewal theorem). Consider a recurrent renewal process S with i.i.d. inter renewal time sequence X having common distribution F and finite mean, associated renewal function m , and a directly Riemann integrable function $z \in \mathbb{D}$.

- (a) If renewal sequence is aperiodic, then $\lim_{t \rightarrow \infty} \int_0^t z_{t-x} dm_x = \frac{1}{\mathbb{E}X_1} \int_0^\infty z_t dt$.
- (b) If renewal sequence has period d , then $\lim_{n \rightarrow \infty} \int_0^{nd} z_{nd-x} dm_x = \frac{d}{\mathbb{E}X_1} \sum_{k \in \mathbb{Z}_+} z_{kd}$.

Proposition 2.2 (Equivalence). Blackwell's theorem and key renewal theorem are equivalent.

Proof. $\Leftarrow$ Let's assume key renewal theorem is true. We fix $a > 0$ and select a simple function $z \in \mathbb{R}_+^{\mathbb{R}_+}$ as an indicator for the interval $[0, a]$, i.e. $z_t \triangleq \mathbb{1}_{[0, a]}(t)$ for any $t \in \mathbb{R}_+$, and $z \in \mathbb{D}$ from Proposition A.4.

- (a) Let S be an aperiodic renewal sequence, then from Key Renewal Theorem, we have $\lim_{t \rightarrow \infty} [m_t - m_{t-a}] = a/\mathbb{E}X_1$.
- (b) Let the period of renewal sequence S be d , then from Key Renewal Theorem, we have for $\lim_{n \rightarrow \infty} [m_{nd-a} - m_{nd}] = d/\mathbb{E}X_1$ for $a < d$. In this case, we have $m_{nd-a} = m_{(n-1)d}$ and the result follows.

$\Rightarrow$. For the converse, we assume that Blackwell's theorem holds true.

- (a) We defer the formal proof of converse for an aperiodic renewal sequence to a later stage. We observe that, from Blackwell theorem, it follows

$$\lim_{t \rightarrow \infty} \frac{dm(t)}{dt} \stackrel{(a)}{=} \lim_{a \rightarrow 0} \lim_{t \rightarrow \infty} \frac{1}{a} (m_{t+a} - m_t) = 1/\mathbb{E}X_1.$$

where in (a) we can exchange the order of limits under certain regularity conditions.

- (b) When renewal sequence has period d , then dm_x is an impulse at multiple of d , and $\int_0^{nd} z_{nd-x} dm_x = \sum_{k=0}^n z_{(n-k)d} (m_{(k+1)d} - m_{kd})$. The result follows from exchange of limits for dRI $z \in \mathbb{D}$.

□

Remark 2. Key renewal theorem is very useful in computing the limiting value of some function g , where g_t is a probability or expectation of an event at an arbitrary time t , for a regenerative process. This value is computed by conditioning on the time of last regeneration prior to time t .

Corollary 2.3 (Delayed key renewal theorem). Consider a recurrent delayed renewal process S with independent inter renewal times X with first inter renewal time distribution G and common inter renewal time distribution F for $(X_n : n \geq 2)$. Let the renewal function be denoted by m^D , $z \in \mathbb{D}$ be a directly Riemann integrable function.

(a) If delayed renewal sequence is aperiodic, then $\lim_{t \rightarrow \infty} \int_0^t z_{t-x} dm_x = \frac{1}{\mathbb{E}X_2} \int_0^\infty z_t dt$.

(b) If delayed renewal sequence has period d , then $\lim_{n \rightarrow \infty} \int_0^{nd} z_{nd-x} dm_x = \frac{d}{\mathbb{E}X_2} \sum_{k \in \mathbb{Z}_+} z_{kd}$.

Remark 3. Any kernel function $t \mapsto K_t \triangleq P\{z_t \in A, X_1 > t\} \leq \bar{F}(t)$, and hence $K \in \mathbb{D}$ from Proposition A.4(b).

A Directly Riemann Integrable

For each scalar $h > 0$ and natural number $n \in \mathbb{N}$, we can define intervals $I_n(h) \triangleq [(n-1)h, nh]$, such that the collection $(I_n(h), n \in \mathbb{N})$ partitions the positive real-line $\mathbb{R}_+$. Consider a function $z \in \mathbb{R}_+ \rightarrow \mathbb{R}_+$ bounded over finite intervals, then we can denote the infimum and supremum of z in the interval I_n as

$$\underline{z}_n^h \triangleq \inf \{z_t : t \in I_n(h)\} \quad \bar{z}_n^h \triangleq \sup \{z_t : t \in I_n(h)\}.$$

We can define functions $\underline{z}^h, \bar{z}^h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\underline{z}_t^h \triangleq \sum_{n \in \mathbb{N}} \underline{z}_n^h \mathbb{1}_{I_n(h)}(t)$ and $\bar{z}_t^h \triangleq \sum_{n \in \mathbb{N}} \bar{z}_n^h \mathbb{1}_{I_n(h)}(t)$ for all $t \in \mathbb{R}_+$. From the definition, we have $\underline{z}^h \leq z \leq \bar{z}^h$ for all $h \geq 0$. The infinite sums of infimum and supremums over all the intervals $(I_n(h), n \in \mathbb{N})$ are denoted by

$$\int_{t \in \mathbb{R}_+} \underline{z}_t^h dt = h \sum_{n \in \mathbb{N}} \underline{z}_n^h, \quad \int_{t \in \mathbb{R}_+} \bar{z}_t^h dt = h \sum_{n \in \mathbb{N}} \bar{z}_n^h.$$

Remark 4. Since $\underline{z}^h \leq z \leq \bar{z}^h$, we observe that $\int_{t \in \mathbb{R}_+} \underline{z}_t^h dt \leq \int_{t \in \mathbb{R}_+} \bar{z}_t^h dt$. For $h = 2^{-k}$, lower bound $\underline{z}^h$ is nondecreasing and upper bound $\bar{z}^h$ is nonincreasing in k . As $h \downarrow 0$, if both left and right limits exist and are equal, then the integral value $\int_{t \in \mathbb{R}_+} z_t dt$ is equal to the limit.

Definition A.1 (directly Riemann integrable (d.R.i.)). A function $z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is directly Riemann integrable and denoted by $z \in \mathbb{D}$ if the partial sums obtained by summing the infimum and supremum of h , taken over intervals obtained by partitioning the positive axis, are finite and both converge to the same limit, for all finite positive interval lengths. That is, $\sum_{n \in \mathbb{N}} h \bar{z}_n^h < \infty$ for all $h > 0$ and $\lim_{h \downarrow 0} \int_{t \in \mathbb{R}_+} \bar{z}_t^h dt = \lim_{h \downarrow 0} \int_{t \in \mathbb{R}_+} \underline{z}_t^h dt$. The limit is denoted by

$$\int_{t \in \mathbb{R}_+} z_t dt = \lim_{h \downarrow 0} \sum_{n \in \mathbb{N}} h \bar{z}_n^h = \lim_{h \downarrow 0} \sum_{n \in \mathbb{N}} h \underline{z}_n^h.$$

For a real function $z \in \mathbb{R}^{\mathbb{R}_+}$, we can define the positive and negative parts by $z^+, z^- \in \mathbb{R}_+^{\mathbb{R}_+}$ such that $z_t^+ \triangleq z_t \vee 0$, and $z_t^- \triangleq -(z_t \wedge 0)$ for all $t \in \mathbb{R}_+$. If both $z^+, z^- \in \mathbb{D}$, then $z \in \mathbb{D}$ and the limit is

$$\int_{\mathbb{R}_+} z_t dt \triangleq \int_{\mathbb{R}_+} z_t^+ dt - \int_{\mathbb{R}_+} z_t^- dt.$$

Remark 5. We compare the definitions of directly Riemann integrable and Riemann integrable functions. For a finite positive M , a function $z \in \mathbb{R}^{[0, M]}$ is Riemann integrable if $\lim_{h \downarrow 0} \int_0^M \underline{z}_t^h dt = \lim_{h \downarrow 0} \int_0^M \bar{z}_t^h dt$. In this case, the limit is the value of the integral $\int_0^M z_t dt$. For a function $z \in \mathbb{R}^{\mathbb{R}_+}$, $\int_{t \in \mathbb{R}_+} z_t dt = \lim_{M \rightarrow \infty} \int_0^M z_t dt$, if the limit exists. For many functions, this limit may not exist.

Remark 6. A directly Riemann integrable function over $\mathbb{R}_+$ is also Riemann integrable, but the converse need not be true.

Lemma A.2. We define a map $z \in \mathbb{R}_+^{\mathbb{R}_+}$ as $z_t \triangleq \sum_{n \in \mathbb{N}} \mathbb{1}_{E_n}(t)$ for all $t \in \mathbb{R}_+$, where $E_n \triangleq \left[n - \frac{1}{2n^2}, n + \frac{1}{2n^2} \right]$ for each $n \in \mathbb{N}$. Then, z is Riemann integrable, but not directly Riemann integrable.

Proof. We will show that z is Riemann integrable, however $\int_{t \in \mathbb{R}_+} \bar{z}_t dt$ is always infinite.

- (a) We observe that $\int_{\mathbb{R}_+} \mathbb{1}_{E_n}(t) dt = 1/n^2$ and hence $\sum_{n \in \mathbb{N}} \int_{\mathbb{R}_+} \mathbb{1}_{E_n}(t) dt = \sum_{n \in \mathbb{N}} 1/n^2 < \infty$. Interchanging infinite sum and integral for positive terms by monotone convergence theorem, we obtain $\int_{t \in \mathbb{R}_+} z_t dt < \infty$, and hence z is Riemann integrable.
- (b) It suffices to show that $\bar{z}_m(h) = 1$ for all $m \in \mathbb{N}$. Since the collection $(I_n(h) : n \in \mathbb{N})$ partitions the entire $\mathbb{R}_+$, for each $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $E_n \cap I_m(h) \neq \emptyset$, and the result follows. $\square$

Exercise A.3 (Necessary conditions for d.R.i.). If a function $z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is directly Riemann integrable, then show that z is bounded and continuous a.e.

Exercise A.4 (Sufficient conditions for d.R.i.). Show that if any of the following conditions hold for a function $z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, then it is directly Riemann integrable.

- (a) z is monotone nonincreasing, and Lebesgue integrable.
- (b) z is bounded above by a directly Riemann integrable function.
- (c) z has bounded support.
- (d) z is continuous, and has finite support.
- (e) z is continuous, bounded, and $\int_{t \in \mathbb{R}_+} \bar{z}_t^h dt$ is bounded for some $h > 0$.
- (f) $\int_{t \in \mathbb{R}_+} \bar{z}_t^h dt$ is bounded for some $h > 0$.

Exercise A.5. For any directly Riemann integrable function $z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ show that $\lim_{t \rightarrow \infty} z_t = \lim_{n \rightarrow \infty} \bar{z}_n^h$.

Proposition A.6 (Tail Property). If $z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is directly Riemann integrable and has bounded integral value, then $\lim_{t \rightarrow \infty} z_t = 0$.

Proof. If $z \in \mathbb{ID}$, then $h \sum_{n \in \mathbb{N}} \bar{z}_n^h < \infty$ for all $h > 0$. This implies that the infinite positive sum $\sum_{n \in \mathbb{N}} \bar{z}_n^h$ is finite, and hence $\lim_{n \rightarrow \infty} \bar{z}_n^h = \lim_{t \rightarrow \infty} z_t = 0$. $\square$

Corollary A.7. Any distribution $F \in \mathcal{M}_+(\mathbb{R}_+)$ with finite mean μ , the complementary distribution function $\bar{F}$ is d.R.i.

Proof. Since $\bar{F}$ is monotonically nonincreasing and its Lebesgue integration is $\int_{\mathbb{R}_+} \bar{F}_t dt = \mu$, the result follows from Proposition A.4(a). $\square$