Lecture-12: Limiting marginal distribution

1 Applications of key renewal theorem

Definition 1.1. Consider a delayed regenerative process $Z : \Omega \to \mathcal{Z}^{\mathbb{R}_+}$ over a delayed renewal sequence S with independent inter renewal time sequence X with first inter renewal time distribution G and subsequent inter renewal times distribution F. For a Borel measurable set $A \in \mathcal{B}(\mathcal{Z})$, we define marginal distribution $f \in [0,1]^{\mathbb{R}_+}$ and kernel functions $K^1, K^2 \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as

$$f_t \triangleq P\{Z_t \in A\}, \qquad K_t^1 \triangleq P\{Z_t \in A, X_1 > t\}, \qquad K_t^2 \triangleq P\{Z_{S_1+t} \in A, X_2 > t\}.$$

Theorem 1.2 (Limiting distribution of regenerative process). *Consider a delayed regenerative process Z over a delayed renewal sequence S with independent inter renewal time sequence X and kernel functions K^1, K^2. If $\mathbb{E}X_1, \mathbb{E}X_2$ are finite and Z is aperiodic, then*

$$\lim_{t\to\infty} P\{Z_t \in A\} = \frac{1}{\mathbb{E}X_2}\int_{t=0}^{\infty} K_t^2 dt.$$

Proof. Since $\bar{F}$ is monotone nonincreasing and Lebesgue integrable with integral $\int_{t\in\mathbb{R}_+} \bar{F}_t dt = \mathbb{E}X_2 < \infty$, we have $\bar{F} \in \mathbb{D}$. From the definition of kernel function K^2, we obtain that $K_t^2 \leqslant \bar{F}_t$ for any $A \in \mathcal{B}(\mathcal{Z})$ and $t \in \mathbb{R}_+$. Since $K^2 \leqslant \bar{F}$ and $\bar{F} \in \mathbb{D}$, it follows that $K^2 \in \mathbb{D}$. For delayed renewal sequence S, we denote the associated delayed renewal function by m^D, recall that $f = K^1 + K^2 * m^D$ and observe that $0 \leqslant K^1 \leqslant \bar{G}$. Applying Key renewal theorem to delayed regenerative process Z and observing that $\lim_{t\to\infty} K_t^1 = 0$, we get the limiting probability of the event $\{Z_t \in A\}$ as $\lim_{t\to\infty} f_t = \lim_{t\to\infty}(m^D * K^2)_t = \frac{1}{\mathbb{E}X_2}\int_{t=0}^{\infty} K_t^2 dt$. ☐

2 Age-dependent branching process

Definition 2.1 (Age-dependent branching process). Consider a population, where each organism $i \in \mathbb{N}$ lives for an *i.i.d.* random time period T_i with common distribution function $F \in \mathcal{M}_+(\mathbb{R}_+)$. Just before dying, each organism i produces an *i.i.d.* random number of offsprings J_i with common distribution $p \in \mathcal{M}(\mathbb{Z}_+)$ and mean $n \triangleq \mathbb{E}[J_1] = \sum_{j\in\mathbb{Z}_+} jp_j$. We fix time $t \in \mathbb{R}_+$, and we denote the number of organisms alive at this time t as X_t and its mean as $m_t \triangleq \mathbb{E}X_t$. The stochastic process $X : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is called an *age dependent branching process*. We denote the history of this process until time t by $\mathcal{F}_t$.

Remark 1. This is a popular model in biology for population growth of various organisms. We are interested in computing the evolution of mean population m.

Lemma 2.2. *Consider the age dependent branching process X from Definition 2.1 starting from a single organism. The organism population regenerates into a random number of replicas at the end of life of each organism.*

Proof. Lifetime of each organism is an *i.i.d.* random variable and the number of their offsprings are *i.i.d.* . Hence, at the end of life of each organism, a probabilistic replica of the population is created. ☐

Definition 2.3. For age dependent branching process with *i.i.d.* lifetime distribution for each organism denoted by $F \in \mathcal{M}_+(\mathbb{R}_+)$ and mean number of offsprings n, we define α to be the unique solution to the equation

$$1 = n\int_0^{\infty} e^{-\alpha t} dF(t). \tag{1}$$

In terms of α and F, we define a distribution $G \in \mathcal{M}_+(R_+)$ such that $dG(t) \triangleq ne^{-\alpha t}dF(t)$ for each $t \in \mathbb{R}_+$. We define the inter renewal time distribution $G_1 \triangleq G$ to inductively define the nth renewal time distribution $G_n \triangleq (G_{n-1} * G) \in \mathcal{M}_+(\mathbb{R}_+)$ and the associated renewal function $m^G \triangleq \sum_{n\in\mathbb{N}} G_n$.

Definition 2.4. For age dependent branching process X with mean m and common organism lifetime distribution F from Definition 2.1 and positive scalar α from Definition 2.3, we define a map $f \in [0,1]^{\mathbb{R}_+}$ and kernel $K \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$ as

$$f_t \triangleq e^{-\alpha t} m_t, \qquad\qquad K_t \triangleq e^{-\alpha t} \bar{F}(t).$$

Theorem 2.5. *Consider the age dependent branching process X from Definition 2.1 with initial population $X_0 \triangleq 1$, distribution G and associated renewal function m^G from Definition 2.3, and maps f, K from Definition 2.4. If $\int_{t \in \mathbb{R}_+} t dG(t) \in (0, \infty)$, then $f = (1 + m^G) * K$.*

Proof. Recall that $X_0 = 1$. Let T_1 and J_1 denote the life time and number of offsprings of the first organism. If $T_1 > t$, then X_t is still equal to $X_0 = 1$. In this case, we have

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}} \mid \mathcal{F}_{T_1}] = X_0 \mathbb{1}_{\{T_1 > t\}}. \tag{2}$$

If $T_1 \leqslant t$, then $X_{T_1} = J_1$ and each of the offsprings start the population growth, independent of the past, and stochastically identical to the population growth of the original organism starting at time T_1. That is, $(X^i_{T_1+u}, u \geqslant 0)$ is a stochastic replica of $(X_u, u \geqslant 0)$, and independent for each $i \in [J_1]$. Hence,

$$\mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}} \mid \mathcal{F}_{T_1}] = \mathbb{E}[\sum_{i=1}^{J_1} X^i_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}} \mid \sigma(T_1)] = n m_{t-T_1} \mathbb{1}_{\{T_1 \leqslant t\}}. \tag{3}$$

Combining the case of number of organisms alive before first birth $\{T_1 > t\}$ from (2), and the case of number of organisms alive after first birth $\{T_1 \leqslant t\}$ from (3), we can write the mean function m_t as

$$m_t = \mathbb{E}[X_t \mathbb{1}_{\{T_1 > t\}}] + \mathbb{E}[X_t \mathbb{1}_{\{T_1 \leqslant t\}}] = \bar{F}(t) + n \int_0^t m_{t-u} dF(u). \tag{4}$$

This looks almost like a renewal function. We take positive scalar α and distribution G from Definition 2.3. Multiplying both sides of (4) by $e^{-\alpha t}$, substituting the definition of G, and substituting the definition of f, K from Definition 2.4, we get for each $t \in \mathbb{R}_+$

$$f_t = m_t e^{-\alpha t} = e^{-\alpha t} \bar{F}(t) + n \int_0^t e^{-\alpha(t-u)} m_{t-u} e^{-\alpha u} dF(u) = K_t + \int_0^t f_{t-u} dG_u.$$

That is, we can rewrite (4) as a renewal equation $f = K + f * G$ for map f in terms of kernel K and inter renewal distribution G. The result is the unique solution of this renewal equation. □

Corollary 2.6. *Consider the age dependent branching process X from Definition 2.1 with initial population $X_0 = 1$ and mean number of offspring n and organism lifetime distribution F. Let positive scalar α and G be from Definition 2.3. If $\int_{t \in \mathbb{R}_+} t dG(t) < \infty$, then $\lim_{t \to \infty} m_t e^{-\alpha t} = \frac{\frac{1}{\alpha}(1 - \frac{1}{n})}{\int_0^\infty t dG(t)}$.*

Proof. From Theorem 2.5, we know that $f = (1 + m^G) * K$ where map $f_t = m_t e^{-\alpha t}$ and kernel function $K_t = e^{-\alpha t} \bar{F}(t)$ are defined for each $t \in \mathbb{R}_+$ in Definition 2.4, and m^G is the renewal function for inter renewal distribution G. Since $\bar{F}$ is monotone non increasing and Lebesgue integrable, it is directly Riemann integrable. From the definition of kernel function K and positivity of scalar α, we have $0 \leqslant K \leqslant \bar{F}$, and hence is directly Riemann integrable and $\lim_{t \to \infty} K_t = 0$. Hence, we can apply key renewal theorem to f to obtain the following limit

$$\lim_{t \to \infty} f_t = \lim_{t \to \infty} m_t e^{-\alpha t} = \frac{\int_{t \in \mathbb{R}_+} K_t dt}{\int_{t \in \mathbb{R}_+} t dG(t)}.$$

From the definition of kernel K, integration by parts, and definition of scalar α in (1), we can write the numerator as $\int_0^\infty e^{-\alpha t} \bar{F}(t) dt = \frac{1}{\alpha} - \frac{1}{\alpha} \int_0^\infty e^{-\alpha t} dF(t) = \frac{1}{\alpha}\left(1 - \frac{1}{n}\right)$. □

Remark 2. We conclude that the mean population is growing exponentially asymptotically, and $m_t \approx Ce^{\alpha t}$ for large time t, some constant C independent of time t, and a positive exponent α.

3 Inspection Paradox

Definition 3.1. Consider a renewal sequence S with *i.i.d.* inter renewal time sequence X and associated counting process N. At each time $t \in \mathbb{R}_+$, we define the current renewal interval length as $X_{N_t+1} \triangleq Y_t + A_t$, its complementary distribution as $g_t^x \triangleq P\{X_{N_t+1} > x\}$, and kernel map $k_t^x \triangleq P\{X_{N_t+1} > x, S_1 > t\}$ for each $x \in \mathbb{R}_+$.

Lemma 3.2 (Inspection Paradox). *Consider a renewal sequence S with* i.i.d. *inter renewal time sequence X and associated counting process N. If $\mathbb{E}X_1 \in (0,\infty)$, then $\mathbb{E}X_{N_t+1} \geqslant \mathbb{E}X_1$.*

Proof. From the definition of expectation of positive random variables, it suffices to show that $g_t^x \geqslant \bar{F}(x)$ for each $x, t \in \mathbb{R}_+$. We will show this in steps.

Step 1. We first observe that X_{N_t+1} is a regenerative process with regeneration instant sequence S since its segment during the nth renewal period $[S_{n-1}, S_n)$ is $\xi_n \triangleq (X_n, (X_n, t \in [S_{n-1}, S_n)))$.

Step 2. From the definition of kernel function, we have $k_t^x = \bar{F}(x \vee t)$ for each $t, x \in \mathbb{R}_+$. From the regenerative property of the length of the current renewal period, we obtain the renewal equation $g_t^x = k_t^x + \mathbb{E}g_{t-S_1}^x \mathbb{1}_{\{S_1 \leqslant t\}}$, i.e. $g^x = k^x + g^x * F$. Applying renewal theorem for finite mean inter renewal times X, we can write the unique solution to the renewal equation as $g^x = (1+m) * k^x$.

Step 3. We define nondecreasing functions $z \mapsto f(z) \triangleq \mathbb{1}_{\{z>x\}}$ and $z \mapsto g(z) \triangleq \mathbb{1}_{\{z>t\}}$. Applying the Chebyshev's inequality to f, g, and random variable X_1, we obtain

$$k_t^x = \mathbb{E}\mathbb{1}_{\{X_{N_t+1}>x, X_1>t\}} = \mathbb{E}\mathbb{1}_{\{X_1>x, X_1>t\}} \geqslant \bar{F}(x)\bar{F}(t).$$

Taking convolution with $1+m$ on both sides of the above equation, and observing that $g^x = (1+m) * k^x$ and $(1+m) * \bar{F} = 1$, we get the result.

□

Remark 3. The accumulated reward R_{N_t+1} in the current renewal interval is possibly dependent on the current renewal duration X_{N_t+1}. If the reward accrual rate is positive, then it follows from the inspection paradox that $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.

Definition 3.3. Consider an *i.i.d.* renewal reward sequence (X, R) with renewal sequence S and counting process N. At each time $t \in \mathbb{R}_+$, we define the marginal tail probability of next reward as $f_t^x \triangleq P\{R_{N_t+1} > x\}$ and associated kernel map $k_t^x \triangleq P\{R_{N_t+1} > x, S_1 > t\}$ for each $x \in \mathbb{R}_+$.

Lemma 3.4. *For a renewal reward process with positive reward accrual rate, we have $\mathbb{E}R_{N_t+1} \geqslant \mathbb{E}R_1$.*

Proof. From the definition of expectation of positive random variables, it suffices to show that $f_t^x \geqslant P\{R_1 > x\}$ for each $x, t \in \mathbb{R}_+$. We will show this in steps.

Step 1. We recall that R_{N_t+1} is a regenerative process with regeneration instant sequence S. From the definition of kernel function, we have $k_t^x = P\{R_1 > x, X_1 > t\}$ for each $t, x \in \mathbb{R}_+$. From the regenerative property of the length of the current renewal period, we obtain the renewal equation $g_t^x = k_t^x + \mathbb{E}g_{t-S_1}^x \mathbb{1}_{\{S_1 \leqslant t\}}$, i.e. $g^x = k^x + g^x * F$. Applying renewal theorem for finite mean inter renewal times X, we can write the unique solution to the renewal equation as $g^x = (1+m) * k^x$.

Step 2. We define nondecreasing functions $z \mapsto f(z) \triangleq \mathbb{1}_{\{z>x\}}$ and $z \mapsto g(z) \triangleq \mathbb{1}_{\{z>t\}}$. Applying the Chebyshev's inequality to f, g, and random variable X_1, we obtain

$$k_t^x = \mathbb{E}\mathbb{1}_{\{R_{N_t+1}>x, X_1>t\}} = \mathbb{E}\mathbb{1}_{\{R_1>x, X_1>t\}} \geqslant P\{R_1 > x\}\bar{F}(t).$$

Taking convolution with $1+m$ on both sides of the above equation, and observing that $g^x = (1+m) * k^x$ and $(1+m) * \bar{F} = 1$, we get the result.

□

A Chebyshev's sum inequality

Theorem A.1. *Consider two monotone measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ either both nonincreasing or nondecreasing, and a random variable $X : \Omega \to \mathbb{R}$. Then, $\mathbb{E}f(X)g(X) \geqslant \mathbb{E}f(X)\mathbb{E}g(X)$.*

Proof. Consider a random vector $Y : \Omega \to \mathbb{R}^2$ to be an *i.i.d.* replica of $X : \Omega \to \mathbb{R}$ and the product $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))$. From the linearity of expectation and Y_1, Y_2 being *i.i.d.* to X, we can expand the mean of the product as

$$\mathbb{E}(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) = 2\mathbb{E}(f(X) - \mathbb{E}f(X))(g(X) - \mathbb{E}g(X)). \quad (5)$$

Since f, g are monotone and both nonincreasing or nondecreasing, we have $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 \geqslant Y_2\}} \geqslant 0$ and $(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2))\mathbb{1}_{\{Y_1 < Y_2\}} \geqslant 0$. Summing the two terms, we obtain

$$(f(Y_1) - f(Y_2))(g(Y_1) - g(Y_2)) \geqslant 0.$$

The result follows by taking expectation on both sides, monotonicity of expectation, and equality in (5). $\square$