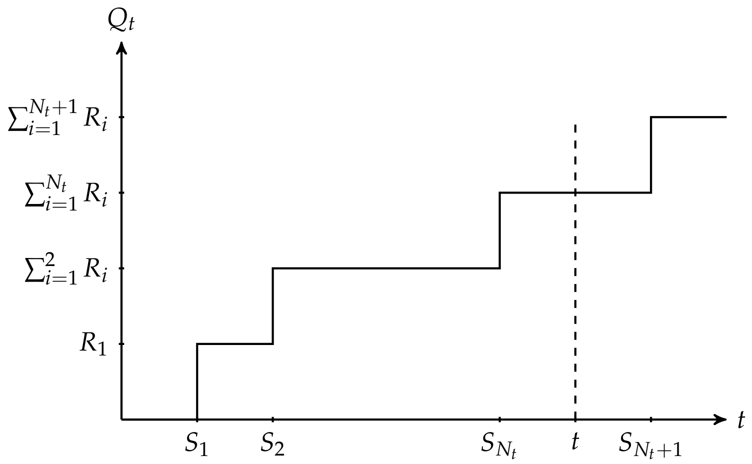
Lecture-13: Renewal reward process

1 Renewal reward process

Definition 1.1 (Renewal reward process). Consider a renewal sequence S with *i.i.d.* inter renewal times X having common distribution F, and the associated counting process N. At the end of each renewal interval $n \in \mathbb{N}$, a random reward R_n is earned at time S_n, where the reward R_n is possibly dependent on the duration X_n, but is *i.i.d.* across intervals $n \in \mathbb{N}$. We assume $(X, R) : \Omega \to (\mathbb{R}_+ \times \mathbb{R})^{\mathbb{N}}$ to be an *i.i.d.* sequence, then R is called *reward sequence* associated with renewal sequence S, and the *reward process* $Q : \Omega \to \mathbb{R}^{\mathbb{R}_+}$ is defined as the accumulated reward earned until each time $t \in \mathbb{R}_+$, denoted as $Q_t \triangleq \sum_{i=1}^{N_t} R_i$. The reward rate at time $t \in \mathbb{R}_+$ is defined as $\frac{Q_t}{t}$.

Example 1.2. Consider a renewal sequence S with *i.i.d.* inter-renewal time sequence X, and associated counting process N. Consider an *i.i.d.* renewal sequence R defined as $R_n \triangleq 1$ for each $n \in \mathbb{N}$. Then the reward process Q is the same as the counting process N.

Example 1.3. Consider a renewal sequence S with *i.i.d.* inter-renewal time sequence X. Consider an *i.i.d.* renewal sequence R defined as $R_n \triangleq X_n$ for each $n \in \mathbb{N}$. Then the reward process Q is the last renewal instant $Q_t = S_{N_t}$ at any time $t \in \mathbb{R}_+$.



Definition 1.4 (Next reward process). For a renewal reward sequence (X, R), we define the reward to earn at next renewal instant as R_{N_t+1}, its mean as $g_t \triangleq \mathbb{E}R_{N_t+1}$, and kernel as $K_t \triangleq \mathbb{E}R_{N_t+1}\mathbb{1}_{\{X_1>t\}}$ at each time $t \in \mathbb{R}_+$.

Lemma 1.5 (Next reward). *Consider a renewal reward sequence (X, R) with renewal sequence S, inter renewal time distribution F, renewal function m, and the associated next reward process. If $\mathbb{E}X_1 \in (0, \infty)$ and $\mathbb{E}|R_1| < \infty$, then the mean g and kernel K of next reward process are related as $g = (1 + m) * K$.*

Proof. The nth segment for next reward process is $\zeta_n \triangleq (X_n, (R_n, t \in [0, X_n)))$. It follows that the segment sequence $\zeta : \Omega \to (\mathbb{R}_+ \times \mathbb{R}^{\mathbb{R}_+})^{\mathbb{N}}$ is *i.i.d.* , and hence R_{N_t+1} is regenerative process with regeneration intervals being the renewal intervals $[S_{n-1}, S_n)$. Considering the event of no renewal or at least one renewal before time t for the regenerative process R_{N_t+1}, we can write at each $t \in \mathbb{R}_+$

$$g_t = \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1>t\}}] + \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}}], \qquad K_t \triangleq \mathbb{E}[R_1\mathbb{1}_{\{S_1>t\}}].$$

From the regenerative property of R_{N_t+1}, we obtain $\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{1}_{\{S_1\leqslant t\}}g_{t-S_1}$. Together with the tower property of conditional expectation, we have at each $t \in \mathbb{R}_+$

$$\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}}] = \mathbb{E}[\mathbb{E}[R_{N_t+1}\mathbb{1}_{\{S_1\leqslant t\}} \mid \mathcal{F}_{S_1}] = \mathbb{E}[\mathbb{1}_{\{S_1\leqslant t\}}g_{t-S_1}].$$

Combining the two cases, we get the renewal equation $g = K + g * F$. Applying renewal theorem to this renewal equation for g with kernel K and inter renewal time distribution F, we obtain the result. □

Corollary 1.6. *If* $\mathbb{E}X_1 \in (0,\infty)$ *and* $\mathbb{E}|R_1| < \infty$, *then* $\lim_{t\to\infty} \frac{1}{t}g_t = 0$.

Proof. From Lemma 1.5, we have $g = (1+m) * K$ where m is the renewal function and K is the kernel function. Applying the conditional Jensen's inequality to convex function absolute, we observe that the kernel function K is bounded above as

$$K_t \triangleq \mathbb{E}[R_{N_t+1}\mathbb{1}_{\{X_1>t\}}] = \mathbb{E}[\mathbb{E}[R_1\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]] \leqslant \mathbb{E}[\mathbb{E}[|R_1|\,\mathbb{1}_{\{X_1>t\}} \mid \sigma(X_1)]].$$

From finiteness of $\mathbb{E}|R_1|$, it follows that $\lim_{t\to\infty} K_t = 0$. We fix $\epsilon > 0$ and select T such that $|K_u| \leqslant \epsilon$ for all $u \geqslant T$. Since $g = (1+m) * K$, we apply triangle inequality to obtain for all $t \geqslant T$,

$$\frac{|g_t|}{t} \leqslant \frac{|K_t|}{t} + \int_0^{t-T} \frac{|K_{t-u}|}{t} dm_u + \int_{t-T}^{t} \frac{|K_{t-u}|}{t} dm_u \leqslant \frac{\epsilon}{t} + \frac{\epsilon m_{t-T}}{t} + \mathbb{E}|R_1|\frac{(m_t - m_{t-T})}{t}.$$

Taking limits and applying elementary renewal and Blackwell's theorem, we get $\limsup_{t\to\infty} \frac{|g_t|}{t} \leqslant \frac{\epsilon}{\mathbb{E}X_1}$. The result follows since $\epsilon > 0$ was arbitrary. □

Theorem 1.7 (Renewal reward). *Consider a renewal reward sequence* (X,R) *and the associated reward process* Q. *If* $\mathbb{E}X_1 \in (0,\infty)$ *and* $\mathbb{E}|R_1| < \infty$, *then the reward rate converges almost surely and in mean, i.e.*

$$\lim_{t\to\infty} \frac{1}{t}Q_t = \frac{\mathbb{E}R_1}{\mathbb{E}X_1} \text{ a.s. }, \qquad \lim_{t\to\infty} \frac{1}{t}\mathbb{E}Q_t = \frac{\mathbb{E}R_1}{\mathbb{E}X_1}.$$

Proof. We will first show the almost sure convergence and then the convergence in mean.

(a) From the strong law of large numbers and strong law for counting processes, we have the following almost sure equalities

$$\lim_{t\to\infty} \frac{Q_t}{N_t} = \lim_{t\to\infty} \frac{1}{N_t}\sum_{i=1}^{N_t} R_i = \mathbb{E}R_1, \qquad \lim_{t\to\infty} \frac{1}{t}N_t = \frac{1}{\mathbb{E}X_1}.$$

The almost sure convergence follows from writing the reward rate as $\frac{1}{t}Q_t = (Q_t/N_t)(N_t/t)$.

(b) Let $\mathcal{F}_\bullet$ be the natural filtration associated with the renewal reward sequence such that for each $n \in \mathbb{N}$, we have $\mathcal{G}_n \triangleq \sigma(X_1, R_1, \dots, X_n, R_n)$. Then $N_t + 1$ is a stopping time adapted to $\mathcal{G}_\bullet$ where almost sure finiteness follows from the fact that $\mathbb{E}X_1 > 0$ and adaptation is clear. Hence, it follows from Wald's lemma and the definition of mean of next reward process, that

$$\mathbb{E}Q_t = \mathbb{E}\sum_{i=1}^{N_t} R_i = \mathbb{E}\sum_{i=1}^{N_t+1} R_i - \mathbb{E}R_{N_t+1} = (m_t+1)\mathbb{E}R_1 - g_t.$$

The result follows by dividing both sides of the equation by $t > 0$, taking limit $t \to \infty$, and applying elementary renewal theorem and Corollary 1.6.

□

Corollary 1.8. *Renewal reward theorem applies to a reward process* Q *that accrues positive reward continuously with a rate* $U : \Omega \to \mathbb{R}_+^{\mathbb{R}_+}$ *such that* $Q_t \triangleq \int_{s=0}^t U_s ds$ *and* $R_n \triangleq Q_{S_n} - Q_{S_{n-1}}$ *is the positive reward accrued in renewal duration* $(S_{n-1}, S_n]$, *with the sequence*(X,R) *being* i.i.d. .

Proof. From positivity of reward accrual rate, it follows that $\sum_{n=1}^{N_t} R_n \leqslant Q_t \leqslant \sum_{n=1}^{N_t+1} R_n$. Dividing the above equation by positive t, almost sure convergence of reward rate follows from application of strong law of large numbers. For the convergence in mean, we observe that at each $t \in \mathbb{R}_+$

$$(m_t+1)\mathbb{E}R_1 - \mathbb{E}R_{N_t+1} \leqslant \mathbb{E}Q_t \leqslant (m_t+1)\mathbb{E}R_1.$$

The second result follows from Corollary 1.6 that shows $\lim_{t\to\infty} \frac{1}{t}\mathbb{E}R_{N_t+1} = 0$. □

2 Alternating renewal processes

Definition 2.1 (Alternating renewal sequence). An *i.i.d.* random sequence $(Z,Y) : \Omega \to (\mathbb{R}_+^2)^{\mathbb{N}}$ is called an *alternating renewal sequence*, where Z_n and Y_n are nth *on* and *off* durations respectively. The on time duration Z_n and off time duration Y_n are not necessarily independent. We define *i.i.d.* inter renewal sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and corresponding renewal sequence $S : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ as $X_n \triangleq Z_n + Y_n$ and $S_n \triangleq \sum_{k=1}^{n} X_k$ for each $n \in \mathbb{N}$. We denote the associated counting process by N and the age process by A. We call the time interval $(S_{n-1}, S_{n-1} + Z_n]$ as nth on time followed by nth off time $(S_{n-1} + Z_n, S_n]$. We denote the distributions for on, off, and renewal periods by H, G, and F, respectively. When on and off times are independent, $F = H * G$.

Definition 2.2 (Alternating renewal process). For an alternating renewal sequence (Z,Y), we define an alternating stochastic process $W : \Omega \to \{0,1\}^{\mathbb{R}_+}$ such that W_t indicates the renewal process S being in on state at time $t \in \mathbb{R}_+$. In particular, we can write alternating renewal process for any time $t \in \mathbb{R}_+$, as

$$W_t \triangleq \mathbb{1}_{\left\{A_t \leqslant Z_{N_t+1}\right\}}.$$

We define the *on* probability at any time $t \in \mathbb{R}_+$ as $P_t \triangleq P\{W_t = 1\}$.

Remark 1. Alternating renewal processes form an important class of renewal processes, and model many interesting applications.

Lemma 2.3. *Alternating renewal process is a regenerative process.*

Proof. For each $n \in \mathbb{N}$, we observe that $W_{S_{n-1}+t} = \mathbb{1}_{(0,Z_n]}(t)$ for all $t \in (0, X_n]$. Hence, we define the nth segment

$$\zeta_n \triangleq (X_n, (\mathbb{1}_{(0,Z_n]}(t) : t \in (0, X_n])),$$

and observe that the segment sequence $\zeta : \Omega \to (\mathbb{R}_+ \times \{0,1\}^{\mathbb{R}_+})^{\mathbb{N}}$ is *i.i.d.* . Therefore, it follows that the alternating renewal process W is regenerative over renewal sequence S. □

Theorem 2.4. *Consider the alternating renewal process W in Definition 2.2 over renewal sequence S with associated renewal function m, and the distribution of* on *times denoted by H. If* i.i.d. *inter renewal times have finite mean and W is aperiodic, then $P = (1+m) * \bar{H}$.*

Proof. In terms of the complementary distribution function $\bar{H}$ of the on times, we can compute the associated kernel function $K \in [0,1]^{\mathbb{R}_+}$ for each $t \in \mathbb{R}_+$, as

$$K_t \triangleq P\{W_t = 1, S_1 > t\} = P\{H_1 > t\} = \bar{H}_t.$$

Recall that alternating renewal process W is regenerative from Lemma 2.3. The result follows from the application of renewal theorem to regenerative process W with renewal function m for inter renewal times and kernel function K. □

Corollary 2.5. *Consider the alternating renewal sequence (Z,Y) in Definition 2.1 and alternating renewal process W in Definition 2.2. If* i.i.d. *inter renewal times have finite mean and W is aperiodic, then the limiting* on *probability is*

$$\lim_{t\to\infty} P_t = \frac{\mathbb{E}Z_1}{\mathbb{E}Z_1 + \mathbb{E}Y_1}.$$

Proof. Recall that the distribution of on times Z and off times Y are denoted by H and G respectively. From Theorem 2.4, the on probability at time $t \in \mathbb{R}_+$ is $P_t = \bar{H}_t + (m * \bar{H})_t$. Since $\mathbb{E}Z_1 < \infty$, we have $\lim_{t\to\infty} \bar{H}_t = 0$. Further, $\bar{H}$ is a directly Riemann integrable function since it is monotone nonincreasing and Lebesgue integrable to $\int_{t\in\mathbb{R}_+} \bar{H}_t dt = \mathbb{E}Z_1 < \infty$. Since $\mathbb{E}Z_1$ and $\mathbb{E}Y_1$ are finite and W is aperiodic, applying key renewal theorem to the limiting probability of alternating process being on, we get

$$\lim_{t\to\infty} P_t = \lim_{t\to\infty} (m * \bar{H})_t = \frac{1}{\mathbb{E}X_1} \int_{t\in\mathbb{R}_+} \bar{H}_t dt.$$

The result follows since $X_1 = Z_1 + Y_1$ and $\int_{t\in\mathbb{R}_+} \bar{H}_t dt = \mathbb{E}Z_1$. □