

Lecture-21: Invariant distribution of Markov processes

1 Class properties

Definition 1.1. For a time homogeneous continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$, we say a state y is *reachable* from state x if $P_{xy}(t) > 0$ for some $t > 0$, and we denote $x \rightarrow y$. If two states $x, y \in \mathcal{X}$ are reachable from each other, we say that they *communicate* and denote it by $x \leftrightarrow y$.

Lemma 1.2. Communication is an equivalence relation.

Definition 1.3. Communication equivalence relation partitions the state space $\mathcal{X}$ into equivalence classes called *communicating classes*. A continuous time Markov chain with a single communicating class is called *irreducible*.

Theorem 1.4. A pure jump continuous time Markov chain with all stable states and its embedded discrete time Markov chain have the same communicating classes.

Proof. It suffices to show that $x \rightarrow y$ for the regular Markov process iff $x \rightarrow y$ in the embedded chain. If $x \rightarrow y$ for the embedded chain, then there exists a path $x = x_0, x_1, \dots, x_n = y$ such that $p_{x_0 x_1} p_{x_1 x_2} \dots p_{x_{n-1} x_n} > 0$ and $0 < \nu_{x_0} \nu_{x_1} \dots \nu_{x_{n-1}}$. It follows that S_n is sum of n independent exponential random variables with positive rates $\nu_{x_0}, \dots, \nu_{x_{n-1}}$, hence almost surely finite, and a stopping time with respect to natural filtration of X . We can write

$$P_{xy}(t) \geq P\{X_0 = x_0, X_{S_1} = x_1, \dots, X_{S_n} = x_n, N_t = n\} = \prod_{k=0}^{n-1} p_{x_k x_{k+1}} \mathbb{E}[\mathbb{1}_{\{N_t=n\}} \mid \cap_{i=0}^n \{Z_i = x_i\}] > 0.$$

Conversely, if the state y is not reachable from state x in embedded chain, then it won't be reachable in the regular continuous time Markov chain. $\square$

Corollary 1.5. A pure jump continuous time Markov chain with all stable states is irreducible iff its embedded discrete time Markov chain is irreducible.

Remark 1. There is no notion of periodicity in continuous time Markov chains since there is no fundamental time-step that can be used as a reference to define such a notion. In fact, for any state $x \in \mathcal{X}$ of a non-instantaneous homogeneous continuous time Markov chain we have $P_{xx}(t) > e^{-\nu_x t} > 0$ for all $t \geq 0$.

1.1 Recurrence and transience

Consider a continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ and its embedded discrete time Markov chain $Z : \Omega \rightarrow \mathcal{X}^{\mathbb{Z}^+}$.

Definition 1.6. Let $k \in \mathbb{N}$. For any state $x \in \mathcal{X}$, we denote the k th return time to state x by $\tau_x^+(k)$ and k th sojourn time in state x by $Y_k^{(x)}$. We inductively define $\tau_x^+(0) \triangleq 0$ and

$$\tau_x^+(k) \triangleq \inf \left\{ t > \tau_x^+(k-1) + Y_k^{(x)} : X_t = x \right\}.$$

Definition 1.7. A state $x \in \mathcal{X}$ is said to be *transient* if $P_x \{\tau_x^+(1) < \infty\} < 1$ and *recurrent* if $P_x \{\tau_x^+(1) < \infty\} = 1$. Furthermore, a recurrent state x is said to be *null recurrent* if $\mathbb{E}_x \tau_x^+(1) = \infty$ and *positive recurrent* if $\mathbb{E}_x \tau_x^+(1) < \infty$.

Definition 1.8. We denote the number of visits to state y during k th successive visit to state x by

$$N_{xy}(k) \triangleq \sum_{n \in \mathbb{Z}_+} \mathbb{1}_{\{[S_n, S_{n+1}) \subseteq [\tau_x^+(k-1), \tau_x^+(k))\}} \mathbb{1}_{\{Z_n = y\}}.$$

The total number of visits to all states during k th successive visit to state x is defined as

$$N_x(k) \triangleq \sum_{y \in \mathcal{X}} N_{xy}(k) = \sum_{n \in \mathbb{Z}_+} \mathbb{1}_{\{[S_n, S_{n+1}] \subseteq [\tau_x^+(k-1), \tau_x^+(k)]\}} = N_{\tau_x^+(k)} - N_{\tau_x^+(k-1)}.$$

The total number to visits to all states before k th return to state x is defined as $N_{\tau_x^+(k)} \triangleq \sum_{j=1}^k N_x(j)$.

Lemma 1.9. We define the j th sojourn time in state y during k th return duration $[\tau_x^+(k-1), \tau_x^+(k)]$ for state x as $Y_{kj}^{(y)}$. Then, the k return time to state x is $\tau_x^+(k) = \tau_x^+(k-1) + \sum_{y \in \mathcal{X}} \sum_{j=1}^{N_{xy}(k)} Y_{kj}^{(y)}$.

Proof. Since $1 = \mathbb{1}_{\{X_t \in \mathcal{X}\}} = \mathbb{1}_{\cup_{y \in \mathcal{X}} \{X_t = y\}} = \sum_{y \in \mathcal{X}} \mathbb{1}_{\{X_t = y\}}$, we can write the following equality

$$\tau_x^+(k) = \tau_x^+(k-1) + \int_{\tau_x^+(k-1)}^{\tau_x^+(k)} \sum_{y \in \mathcal{X}} \mathbb{1}_{\{X_t = y\}} dt.$$

Further, we can write $\mathbb{1}_{\{X_t = y\}} = \sum_{n \in \mathbb{Z}_+} \mathbb{1}_{\{Z_n = y\}} \mathbb{1}_{[S_n, S_{n+1})}(t)$. Interchanging sum and integral using monotone convergence theorem, we obtain

$$\tau_x^+(k) = \tau_x^+(k-1) + \sum_{y \in \mathcal{X}} \sum_{n \in \mathbb{Z}_+} (S_{n+1} - S_n) \mathbb{1}_{\{Z_n = y\}} \mathbb{1}_{\{[S_n, S_{n+1}] \subseteq [\tau_x^+(k-1), \tau_x^+(k)]\}}.$$

We observe that $\{n \in \mathbb{Z}_+ : N_{\tau_x^+(k-1)} \leq n < N_{\tau_x^+(k)}\} = \{n \in \mathbb{Z}_+ : [S_n, S_{n+1}] \subseteq [\tau_x^+(k-1), \tau_x^+(k)]\}$, and hence $V_{xy}(k) \triangleq \{n \in \mathbb{Z}_+ : N_{\tau_x^+(k-1)} \leq n < N_{\tau_x^+(k)}, Z_n = y\}$ is the set of transitions which correspond to visits to state y during k th return time to state x , and $N_{xy}(k) = |V_{xy}(k)|$. Further, the duration $S_{n+1} - S_n$ is the sojourn time in state Z_n . Therefore, the result follows. $\square$

Theorem 1.10. Consider a pure jump homogeneous continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with embedded discrete time Markov chain $Z : \Omega \rightarrow \mathcal{X}^{\mathbb{Z}_+}$. If Z is recurrent then all states of X are stable, and the number of jumps is finite in any finite time t .

Proof. Since X is a pure jump Markov process, transition rate $\nu_x < \infty$ for each state $x \in \mathcal{X}$. Let $X_0 = x \in \mathcal{X}$ be the initial state. Let $N(x, n) \triangleq \sum_{k=1}^n \mathbb{1}_{\{Z_k = x\}}$ be the number of visits to a state $x \in \mathcal{X}$ in the first n transitions and T_i^x be the i th sojourn time in the state x . From the recurrence of the embedded chain, the state x occurs infinitely often, i.e. $\lim_{n \in \mathbb{N}} N(x, n) = \infty$ almost surely. It follows that the sojourn time sequence $T^x : \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ is i.i.d. and exponentially distributed with mean $\mathbb{E}T_i^x = 1/\nu_x < \infty$ and hence $\nu_x > 0$. It follows that $\nu_x \in (0, \infty)$ and the state x is stable. Since the choice of x is arbitrary, it follows that each state $x \in \mathcal{X}$ is stable.

Since $S_n \geq \sum_{i=1}^{N(x, n)} T_i^x$, we get that $m_t = \sum_{n \in \mathbb{N}} P\{S_n \leq t\} \leq \sum_{n \in \mathbb{N}} P\left\{\sum_{i=1}^{N(x, n)} T_i^x \leq t\right\} = \nu_x t$. It follows that N_t is almost surely finite for any finite time $t \in \mathbb{R}_+$. $\square$

Theorem 1.11. Consider a pure jump irreducible continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with embedded discrete time Markov chain $Z : \Omega \rightarrow \mathcal{X}^{\mathbb{Z}_+}$. X is recurrent iff Z is recurrent.

Proof. There is nothing to prove for $|\mathcal{X}| = 1$. Hence, we assume $|\mathcal{X}| \geq 2$ without loss of generality.

(a) We assume that the embedded Markov chain is recurrent, then each state is stable from Theorem 1.10. Further, a pure jump continuous time Markov chain with all stable states is irreducible iff embedded discrete time Markov chain is irreducible from Corollary 1.5. Let $X_0 = x \in \mathcal{X}$, and we observe that $N_{\tau_x^+(1)}$ is a finite random variable and the recurrence time

$$\tau_x^+(1) = \sum_{n \in \mathbb{Z}_+} (S_{n+1} - S_n) \mathbb{1}_{\{[S_n, S_{n+1}] \subseteq [0, \tau_x^+(1)]\}} = \sum_{n=0}^{\tau_x^+(1)} (S_{n+1} - S_n).$$

is an a.s. finite sum of finite random variables, it follows that $\tau_x^+(1)$ is finite almost surely.

(b) Conversely, if the embedded Markov chain is not recurrent, it has a transient state $x \in \mathcal{X}$ for which $P_x\{N_x(1) = \infty\} > 0$. By the same argument, $P_x\{\tau_x^+(1) = \infty\} > 0$ and hence the continuous time Markov chain is not recurrent. $\square$

Corollary 1.12. Recurrence is a class property.

Theorem 1.13. Consider an irreducible positive recurrent discrete time Markov chain $Z : \Omega \rightarrow \mathcal{X}^{\mathbb{Z}^+}$ with transition probability matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ and invariant distribution $u \in \mathcal{M}(\mathcal{X})$. Then,

$$u_y = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbb{1}_{\{Z_n=y\}} = \frac{\mathbb{E}_x N_{xy}(k)}{\mathbb{E}_x N_x(k)} = u_x \mathbb{E}_x N_{xy}(k).$$

Proof. Let $Z_0 = x$. For a homogeneous Markov chain Z , the random sequence $N_{\tau_x^+} : \Omega \rightarrow \mathbb{N}^{\mathbb{N}}$ is a renewal sequence, and the number of visits $N_x(k)$ to all states before the k th return to state x is the k th inter-return time to state x . The number of visits to state y between two successive visits to state x is

$$N_{xy}(k) = \sum_{n=N_{\tau_x^+}(k-1)}^{N_{\tau_x^+}(k)-1} \mathbb{1}_{\{Z_n=y\}}.$$

We can consider $N_{xy}(k)$ as the reward in the k th renewal duration. The result follows from the renewal reward theorem and the fact that $N_{xx}(k) = 1$ for all $k \in \mathbb{N}$ and $x \in \mathcal{X}$. $\square$

Theorem 1.14. Consider an irreducible recurrent continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with sojourn time rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ and transition matrix $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ for the embedded Markov chain $Z : \Omega \rightarrow \mathcal{X}^{\mathbb{Z}^+}$. Let $u \in \mathbb{R}_+^{\mathcal{X}}$ be any strictly positive solution of $u = up$, then for each state $x \in \mathcal{X}$

$$\mathbb{E}_x \tau_x^+(1) = \frac{1}{u_x} \sum_{y \in \mathcal{X}} \frac{u_y}{\nu_y}. \quad (1)$$

Further, the process X is positive recurrent iff $\sum_{x \in \mathcal{X}} \frac{u_x}{\nu_x} < \infty$.

Proof. Let $X_0 = x \in \mathcal{X}$. Recall that $Y_k^{(x)}$ denotes the k th sojourn time of the Markov process X in state x , and the random sequence $Y^{(x)} : \Omega \rightarrow \mathbb{R}_+^{\mathbb{N}}$ is i.i.d. with common exponential distribution of rate ν_x . From Lemma 1.9, the first visit time to state x in terms of $N_{xy}(1)$ and sojourn times $Y_k^{(y)}$ for each state $y \in \mathcal{X}$, is $\tau_x^+(1) = \sum_{y \in \mathcal{X}} \sum_{k=1}^{N_{xy}(1)} Y_k^{(y)}$. We recall that jump chain Z and sojourn times are independent given the initial state, and hence $N_{xy}(1)$ and $Y^{(y)}$ sequences are independent for each state $y \neq x$. From taking expectations on both sides, exchanging summation and expectations by the application of monotone convergence theorem for positive random variables, we get $\mathbb{E}_x \tau_x^+(1) = \sum_{y \in \mathcal{X}} \mathbb{E} Y_k^{(y)} \mathbb{E}_x N_{xy}$. To show (1), it suffices to show that $\mathbb{E}_x N_{xy}(k) = \frac{u_y}{u_x}$.

The embedded Markov chain Z inherits the irreducibility and recurrence of the Markov process X from Corollary 1.5 and Theorem 1.11. For irreducible and recurrent Markov chain Z with transition matrix p and any strictly positive solution to $u = up$, we have $\mathbb{E}_x N_{xy}(k) = \frac{u_y}{u_x}$ from Theorem 1.13.

Since u is strictly positive, it follows that $\mathbb{E}_x \tau_x^+(1) < \infty$ iff $\sum_{y \in \mathcal{X}} \frac{u_y}{\nu_y} < \infty$. $\square$

Remark 2. For an irreducible regular continuous time Markov chain X , the embedded Markov chain Z is irreducible and recurrent. If Z with transition matrix p is positive recurrent, then there exists a strictly positive solution equilibrium distribution $u \in \mathcal{M}(\mathcal{X})$ such that $u = up$. However, it is possible that rates $\nu \in \mathbb{R}_+^{\mathcal{X}}$ ensure that $\sum_{y \in \mathcal{X}} \frac{u_y}{\nu_y} = \infty$, in which case X is null recurrent.

2 Invariant Distribution

Remark 3. For a homogeneous Markov process $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with probability transition kernel $P : \mathbb{R}_+ \rightarrow \mathcal{M}(\mathcal{X})^{\mathcal{X}}$, we denote the marginal distribution of random variable X_t at time t by $\nu(t) \in \mathcal{M}(\mathcal{X})$, where for each time $t \in \mathbb{R}_+$

$$\nu(t) = \nu(0)P(t).$$

In general, we can write $\nu(s+t) = \nu(s)P(t)$. Hence, if there exists a stationary distribution $\pi \triangleq \lim_{s \rightarrow \infty} \nu(s)$ for this process X , then we would have $\pi = \pi P(t)$ for all times $t \in \mathbb{R}_+$.

Definition 2.1. A distribution $\pi \in \mathcal{M}(\mathcal{X})$ is an *invariant distribution* of a homogeneous continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with probability transition kernel $P : \mathbb{R}_+ \rightarrow \mathcal{M}(\mathcal{X})^{\mathcal{X}}$ if $\pi P(t) = \pi$ for all $t \in \mathbb{R}_+$.

Corollary 2.2. For a homogeneous continuous time Markov chain $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ with generator matrix Q , a distribution $\pi \in \mathcal{M}(\mathcal{X})$ is an invariant distribution iff $\pi Q = 0$.

Proof. Recall that we can write the transition probability matrix $P(t)$ at any time $t \in \mathbb{R}_+$ in terms of generator matrix Q as $P(t) = e^{tQ}$. Using the exponentiation of a matrix, we can write

$$\pi P(t) = \pi e^{tQ} = \pi + \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \pi Q^n, \quad t \in \mathbb{R}_+.$$

Therefore, $\pi Q = 0$ iff π is an equilibrium distribution of the Markov process X . $\square$

Theorem 2.3. Let $X : \Omega \rightarrow \mathcal{X}^{\mathbb{R}^+}$ be an irreducible recurrent homogeneous continuous time Markov chain with probability transition kernel $P : \mathbb{R}_+ \rightarrow \mathcal{M}(\mathcal{X})^{\mathcal{X}}$, the transition rate sequence $v \in \mathbb{R}_+^{\mathcal{X}}$, and the transition matrix for embedded jump chain $p \in \mathcal{M}(\mathcal{X})^{\mathcal{X}}$. Then for all states $x, y \in \mathcal{X}$ the $\lim_{t \rightarrow \infty} P_{xy}(t)$ exists, this limit is independent of the initial state $x \in \mathcal{X}$ and denoted by π_y . Let u be any strictly positive invariant measure such that $u = up$. If $\sum_{x \in \mathcal{X}} \frac{u_x}{v_x} = \infty$, then $\pi_y = 0$ for all $y \in \mathcal{X}$. If $\sum_{x \in \mathcal{X}} \frac{u_x}{v_x} < \infty$ then for all $y \in \mathcal{X}$,

$$\pi_y = \frac{\frac{u_y}{v_y}}{\sum_{x \in \mathcal{X}} \frac{u_x}{v_x}} = \frac{v_y^{-1}}{\mathbb{E}_y \tau_y^+(1)}. \quad (2)$$

Proof. Fix a state $y \in \mathcal{X}$, and define a process $W : \Omega \rightarrow \{0, 1\}^{\mathbb{R}^+}$ such that $W_t = \mathbb{1}_{\{X_t = y\}}$. Then, from the regenerative property of the homogeneous continuous time Markov chain and renewal reward theorem, we have

$$\pi_y \triangleq \lim_{t \rightarrow \infty} P_x \{X_t = y\} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{1}_{\{X_s = y\}} ds = \frac{\mathbb{E}_y Y_k^{(y)}}{\mathbb{E}_y \tau_y^+(k)} = \frac{v_y^{-1}}{\mathbb{E}_y \tau_y^+(1)}.$$

We have considered k th hitting time $\tau_y^+(k)$ to state y as k th renewal instant, and the k th sojourn time in state y $\int_{\tau_y^+(k-1)}^{\tau_y^+(k)} \mathbb{1}_{\{X_s = y\}} ds = Y_k^{(y)}$ as the reward in the k th inter renewal period. $\square$

Remark 4. We observe that $\pi Q = 0$ for distribution $\pi \in \mathcal{M}(\mathcal{X})$ defined in (2), since $u = up$ and $Q_{xy} = v_x p_{xy}$ for all states $x, y \in \mathcal{X}$.